BALKANJM 01 (2013) 44-60

Contents lists available at BALKANJM

BALKAN JOURNAL OF MATHEMATICS

journal homepage: www.balkanjm.com

On the Fuzzy Local Information Function

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ARTICLE INFO

Article history:
Received 28 December 2012
Accepted 25 January 2013
Available online 6 February 2013

Keywords:
Fuzzy topological space
Fuzzy probability measure space
Fuzzy probability measure-preserving transformation
Fuzzy information function
Fuzzy local entropy function

ABSTRACT

In this paper, after giving the definition of the fuzzy dynamical system, we investigate some basic properties of the fuzzy information function. Then we focus on the fuzzy local information function of fuzzy dynamical system. Finally, we prove some fundamental results relating to this function.

1. Introduction

In his paper, Shannon [7] firstly introduced the concept of information function and investigated some properties of this function in terms of non-fuzzy sense. Then, McMillan [6] found some results relating to this function.

In the first place, we defined the fuzzy dynamical system and stated the basic properties of this system. It is known that in information theory, the source of information source is quite. Tok has proved some properties of the fuzzy information function in [9]. Other results of fuzzy information function were investigated by Dumitrescu in [4]. Recently, Tok defined the fuzzy local entropy function and stated some properties of this function in [10].

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It is the aim of this work to define the fuzzy local information function and prove some important ergodic properties of this function.

2. Basic definitions

2.1 Definition. Following Zadeh [12], a pair (X,F) is called a fuzzy set. Where X is an arbitrary non-empty set and $A:X\to[0,1]$ is a membership function. That is, a fuzzy set is characterized by a membership function A from X to the closed unit interval $I=[0,1]$.

Thus, we identify a fuzzy set its membership function A. In this connection, *A(x)*, is interpreted as the degree of membership of a point $x\in X$. The family of all fuzzy sub-sets is called a fuzzy class and will be denoted by *F*.

2.2 Definition The first definition of a fuzzy topological space is due to Chang [2].

According to Chang, a fuzzy topological space is a pair (X,F). Where X is an arbitrary non-empty set and F is a fuzzy class if satisfies the following conditions;

i) $0,1\in F$.

ii) If $U,V\in F$, then $U\wedge V\in F$.

iii) If $U_n\in F, n\in N$, then $\sup_{n\in N} U_n = \bigvee_{n\in N} U_n \in F$.

Every element of *F* is called an open fuzzy set or simply *F*-open fuzzy set. The element of *F* is a *F*-closed fuzzy if and only if its complement is a *F*-open fuzzy set.

Now, we define the fuzzy transformation on a fuzzy topological space as follows.

2.3 Definition Let *X* and *Y* be two fuzzy topological spaces. We consider a transformation from the fuzzy topological space *X* to *Y*. Let *B* be a fuzzy sub set of the fuzzy topological space *Y* with membership function $B(y)$ for $y\in Y$. Then, the inverse image of *B* written as $T^{-1}(B)$ is a fuzzy sub set of the fuzzy topological space *X* whose membership function is defined by $T^{-1}(B(x))=B(T(x))$ for all $x\in X$. Conversely, let *A* be a fuzzy sub set of the fuzzy topological space *X* with membership function *A(x)* for $x\in X$. The image of *A* written as $T(A)$ is a fuzzy subset of the fuzzy topological space *Y* whose membership function is given by

$$T(A)(y) = \begin{cases} \sup\limits_{x \in T^{-1}(y)} \{A(x)\} \; if \; T^{-1}(y) \neq \phi \\ 0 \qquad\qquad otherwise \end{cases}$$

for all $y \in Y$. Where $T^{-1}(y) = \{x \mid T(x) = y\}$. Other properties of this transformation see [2].

2.4 Definition A transformation T from the fuzzy topological space (X, F) to another fuzzy topological space (Y, F_1) is a fuzzy continuous transformation or simply *F*–continuous if and only if the inverse of each F_1-open fuzzy set is a *F*-open fuzzy set.

2.5 Definition Let *F* be a fuzzy class. This fuzzy class *F* is called a fuzzy σ-algebra on *X*, if it satisfies the following conditions;

i) For each constant $\alpha \in [0,1], \alpha \in F$.

ii) If $A \in F$, then $\overline{A} = 1 - A \in F$. Where $\overline{A}$ is a complement of the fuzzy set A defined by $\overline{A(x)} = 1 - A(x)$, for each $x \in X$.

iii) If $A_n \in F, n \in N$, then $\sup\limits_n A_n = V_n A_n \in F$.

In this case, the pair (X, F) is a fuzzy measurable space and the elements of *F* are fuzzy measurable sets. For more details, we refer to [8].

2.6 Definition Let (X, F) and (Y, F_1) be two fuzzy measurable spaces. One says the transformation T from (X, F) to (Y, F_1) is fuzzy measurable, if for each $A \in F_1$, then $T^{-1}(A) \in F$. For more properties of this transformation, see, [8] and [9].

2.7 Definition The family $A_1, A_2, \ldots, A_n$ of fuzzy subsets is called disjoint, if $(\overset{j}{\underset{i=1}{V}} A_i) \wedge A_{j+1} = \phi$ for each $j = 1,2,\ldots,n-1$.

2.8 Definition Let (X, A, μ) be a classical probability measure space. See, [3]. A fuzzy probability measure is a fuzzy measurable mapping μ from the fuzzy measurable space (X, F) to [0,1] defined by $\mu(A) = \int A d\mu$ fulfilling the following conditions;

i) $\mu(\phi) = 0$ and $\mu(X) = 1$.

ii) $\mu(A) \geq 0$ for each $A \in F$.

iii) If $(A_n)_{n\in N}$ is a disjoint sequence of fuzzy sets and $A_n \in F$ for every $n\in N$, then we have;

$$\mu(\overset{\infty}{\underset{n=1}{V}} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \text{ (fuzzy } \sigma\text{-additivity).}$$

The triple (X, F, μ) is a fuzzy probability measure space. The elements of **F** are called fuzzy events. For more details, we refer to [8].

2.9 Definition Let (X, F, μ) and (X, F, μ) be two fuzzy probability measure spaces.

Then,

i) The transformation T from (Y, F_1, μ_1) to (Y, F_1, μ_1) is a fuzzy probability measure preserving transformation if $\mu(T^{-1}(A)) = \mu_1(A)$ for each $A\in F_1$.

ii) The transformation T from (X, F, μ) to (Y, F_1, μ_1) is an invertible fuzzy probability measure preserving transformation if *T* is a fuzzy probability measure preserving bijective transformation and T^{-1} is also a fuzzy probability measure preserving transformation

iii) Let (X, F, μ) be a fuzzy probability measure space. The transformation

$T:(X, F, \mu) \to (X, F, \mu)$ is a fuzzy probability measure-preserving transformation, if *T* is a fuzzy measurable transformation and it fulfills the condition $\mu(T^{-1}(A)) = \mu_1(A)$ for each $A\in F$.

iv) The quadruple (X, F, μ, T) is called a fuzzy dynamical system. One will write briefly (X, T) instead of (X, F, μ, T) for convenience. For more properties of this system, see, [5].

3. Fuzzy information function

3.1 Definition Suppose that (X, T) is a fuzzy dynamical system and *I* is a countable set. A collection $P = \{A_1, A_2, \ldots, A_n, \ldots\}$ of the fuzzy measurable events is called a fuzzy partition if $\sum_{i\in I} A_i(x) = 1$, for all $x\in X$ for each with $A_i \neq \phi$ for each $i\in I$.

3.2 Remark It follows from Definition 3.1, that the collection $P = \{A_1, A_2, \ldots, A_n\}$ of the finite fuzzy measurable events is called a finite fuzzy partition if $\sum_{i=1}^{n} A_i(x) = 1$, for all $x\in X$ for each with $A_i \neq \phi$ for each $i = 1,2,\ldots,n$. Let P and Q be two finite fuzzy partitions. If

$\mu(P \wedge Q) = \mu(P)\mu(Q)$, then P and Q are called independents. For more details, we refer to [9].

3.3 Definition Let $P = \{A_1, A_2, \ldots, A_n\}$ be a finite fuzzy measurable partition of fuzzy dynamical system (X,T). Then if $A_i \in P$, $i = 1,2,\ldots,n$ is an observed event. The information $I_{\mu,f}(P)$ carried by P may be defined as; $I_{\mu,f}(P) = \log \frac{1}{\mu(A_i)} = -\log \mu(A_i)$ and the quantity $I_{\mu,f}(P,x) = -\sum_{i=1}^{n} \chi_{A_i}(x).\log \mu(A_i)$ for each $x \in X$ is called a fuzzy information function. In this paper, all logarithms will be taken to be the natural base $"e"$.

Where χ_{A_i} is the characteristic function of A_i defined by $\chi_{A_i}(x) = \begin{cases} 1 \; if \; x \in A_i \\ 0 \; if \; x \notin A_i \end{cases}$

3.4 Proposition Let P and Q be two finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$ and $I_{\mu,f}(Q,x) < \infty$, for $x \in X$. Then, for all $x \in X$.

i) $I_{\mu,f}(P,x) \geq 0$.

ii) If P_0 is a trivial fuzzy measurable partition, i.e. $P_0 = \{X, \phi\}$ then $I_{\mu,f}(P,x) = 0$.

iii) If $P \subset Q$, then $I_{\mu,f}(P,x) < I_{\mu,f}(Q,x)$.

ıv) $I_{\mu,f}(P \vee Q,x) < I_{\mu,f}(P,x) + I_{\mu,f}(Q,x)$.

v) If T is a fuzzy probability measures preserving transformation, then $I_{\mu,f}(T^{-1}P,x) = I_{\mu,f}(P,Tx)$.

Proof See, The Proposition II-3 of (9).

3.5 Lemma Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that is positive and sub additive. Then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and is equal to $\inf_{n\in N} \frac{1}{n} a_n$.

Proof See, the Theorem 4-9 of [11].

3.6 Theorem If P is a finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$, for each $x \in X$ then, $\lim_{n\to\infty} \frac{1}{n} I_{\mu,f}\left(\bigvee_{i=0}^{n-1} T^{-i}P, x \right)$, for each $x \in X$ exists and is equal to the infimum.

Proof Write $a_n = I_{\mu,f}\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P, x\right)$, for each $x \in X$. Clearly, the sequence $(a_n)_{n\geq 1}$ satisfies the conditions of Lemma 3.5, from the Proposition 3.4 (**i**) and (**iv**). Thus one has only the Lemma 3.5 $\lim_{n\to\infty}\frac{1}{n} I_{\mu,f}\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P, x\right)$, for each $x \in X$ exists and is equal to the infimum.

3.7 Definition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$, for $x \in X$.

Then, the limit function $I_{\mu,f}(T,P,x) = \lim_{n\to\infty}\frac{1}{n} I_{\mu,f}\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P, x\right)$, for each $x \in X$ is called he fuzzy information function of T with respect to the finite fuzzy measurable partition P.

3.8 Proposition Let P and Q be two finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$ and $I_{\mu,f}(Q,x) < \infty$, for $x \in X$.

Then, for all $x \in X$

i) $I_{\mu,f}(T,P,x) \geq 0$ With equality if and only if P is a trivial fuzzy measurable partition

ii) $I_{\mu,f}(T,P,x) \leq I_{\mu,f}(P,x)$

iii) If $P \subset Q$, then $I_{\mu,f}(T,P,x) \leq I_{\mu,f}(T,Q,x)$

iv) $I_{\mu,f}(T,P \vee Q,x) \leq I_{\mu,f}(T,P,x) + I_{\mu,f}(T,Q,x)$

v) If T is a fuzzy probability measures preserving transformation, then $I_{\mu,f}(T,P,x) = I_{\mu,f}(T,TP,Tx)$.

Proof See, [1] and [9].

3.9 Proposition Let P be a finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$, for $x \in X$. Then, for all $x \in X$.

i) $I_{\mu,f}(T^k P,x) = kI_{\mu,f}(T,P,x)$ if $k > 0$.

ii) If T is an invertible fuzzy transformation, and $k \in Z$, then

$I_{\mu,f}(T^k,P,x) = |k| I_{\mu,f}(T,P,x)$.

Proof

i) Let P be a finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$, for $x\in X$. We first prove that if for each $x\in X$ and $k>0$,

$$I_{\mu,f}(T^k,\overset{k-1}{\underset{i=0}{V}}T^{-i},x)=kI_{\mu,f}(T,P,x) \tag{1}$$

This follows since

$$\begin{aligned}I_{\mu,f}(T^k,\overset{n}{\underset{i=0}{V}}T^{-i}P,x)&=\lim_{n\to\infty}\frac{1}{n}I_{\mu,f}(\overset{n-1}{\underset{j=0}{V}}T^{-kj}\ (\overset{k-1}{\underset{i=0}{V}}T^{-i}P),x)\\ &=k\lim_{n\to\infty}\frac{1}{nk}I_{\mu,f}(\overset{nk-1}{\underset{i=0}{V}}T^{-i}P,x)=kI_{\mu,f}(T,P,x)\end{aligned} \tag{2}$$

by the theorem 3.6.

Also, for each $x\in X$ and $k>0$, from the proposition 3.8 (**iii**)

$$I_{\mu,f}(T^kP,x)\le I_{\mu,f}(T^k,\overset{k-1}{\underset{i=0}{V}}T^{-i}P,x)=kI_{\mu,f}(T,P,x) \text{ and } I_{\mu,f}(T^kP,x)\le kI_{\mu,f}(T,P,x) \tag{3}$$

Therefore, one writes the following inequality from the proposition 3.8 (**iii**) for each $x\in X$,

$$I_{\mu,f}(T^k,\overset{k-1}{\underset{i=0}{V}}T^{-i}P,x)=I_{\mu,f}(T,P,x) \tag{4}$$

Since for each $x\in X$,

$$I_{\mu,f}(T^k,\overset{k-1}{\underset{i=0}{V}}T^{-i}P,x)\le kI_{\mu,f}(T,P,x) \tag{5}$$

we obtain thus for each $x\in X$,

$$kI_{\mu,f}(T,P,x)\le I_{\mu,f}(T^kP,x) \tag{6}$$

Hence the result follows from inequalities (3) and (6)

ii) Let P be a finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$,

for $x\in X$, Then it suffices that for each $x\in X$,

$$I_{\mu,f}(T^{-1},P,x)=I_{\mu,f}(T,P,x) \tag{7}$$

but for each $x\in X$,

$$I_{\mu,f}(\overset{n-1}{\underset{i=0}{V}}T^{i}P,x)=I_{\mu,f}(T^{-(n-1)}(\overset{n-1}{\underset{i=0}{V}}T^{i}P),x) \tag{8}$$

Thus, we obtain from the Proposition 3.4 (v), for each $x\in X$

$$I_{\mu,f}(\overset{n-1}{\underset{i=0}{V}}(T^{-1})^{-i}P,x)=I_{\mu,f}(\overset{n-1}{\underset{j=0}{V}}T^{-j}P,x) \tag{9}$$

Dividing the last equality (9) by $n>0$ and taking the limit for $n\to\infty$, we obtain from the theorem 3.6 and definition 3.7,

$$I_{\mu,f}(T^{-1},P,x)=I_{\mu,f}(T,P,x).$$

3.10 Definition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with

$I_{\mu,f}(P,x)<\infty$, for $x\in X$. Then, for all $x\in X$, the quantity $I_{\mu,f}(T,x)=\sup_{P}\left\{I_{\mu,f}(T,P,x)\,\middle|\,\begin{array}{r}P\text{ is a finite fuzzy measurable partition}\\ \text{of }(X,T)\text{ with }I_{\mu,f}(P,x)<\infty\end{array}\right\}$ is called the fuzzy information function of fuzzy dynamical system (X,T). Where the supremum is taken over all finite fuzzy measurable partitions of fuzzy dynamical system (X,T) with the finite fuzzy information functions.

3.11 Proposition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with

$I_{\mu,f}(P,x)<\infty$, for $x\in X$. Then, for all $x\in X$,

i) $I_{\mu,f}(T,x)\geq 0$.

ii) $I_{\mu,f}(Id,x)=0$

iii) $I_{\mu,f}(T^{k}P,x)=kI_{\mu,f}(T,P,x)$ if $k>0$.

iv) If T is an invertible fuzzy transformation, and $k\in Z$, then $I_{\mu,f}(T^{k}P,x)=|k|\,I_{\mu,f}(T,P,x)$.

Proof (i) and (ii) are trivial. (iii) and (iv) are similar to the proofs of the previous proposition 3.9 (i) and (ii).

3.12 Definition Let (X,T) and (Y,S) be two fuzzy dynamical systems. We say that (Y,S) is a fuzzy factor of the fuzzy dynamical system (X,T), if there exist $A\in F$ and $B\in F_1$ such that

i) $\mu(A)=1$ and $\mu_1(B)=1$.

ii) There exists a measure-preserving transformation $\varphi : A \to B$ with $\varphi(T(x)) = S(\varphi(x))$ for all $x\in X$.

3.13 Proposition Let (Y,S) be a fuzzy factor of the fuzzy dynamical system (X,T), then for all $x\in X$ and $y\in Y$, $I_{\mu_1,f}(S,y) \le I_{\mu,f}(T,x)$.

Proof Let φ be a fuzzy probability measure-preserving function. If Q is finite fuzzy measurable partition of the fuzzy factor (Y,S) with $I_{\mu,f_1}(Q,y) < \infty$, for $y\in Y$, then $\varphi^{-1}Q$ is a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(\varphi^{-1}Q,x) < \infty$, for $x\in X$ by the proposition II-2 of (9). Therefore, we have the following equality by the Proposition 3.4 (v), for all $x\in X$ and $y\in Y$,

$$I_{\mu_1,f}(Q,y) = I_{\mu,f}(\varphi^{-1}Q,x) \tag{10}$$

It is easy to see that

$$\varphi^{-1}(\mathop{V}_{i=0}^{n-1} S^{-i}Q) = \mathop{V}_{i=0}^{n-1} T^{-i}(\varphi^{-1}Q) \tag{11}$$

Therefore, we write the following equality, for $y\in Y$,

$$I_{\mu,f}(\varphi^{-1}(\mathop{V}_{i=0}^{n-1} S^{-i}Q),y) = I_{\mu,f}(\mathop{V}_{i=0}^{n-1} T^{-i}(\varphi^{-1}Q),\varphi^{-1}y) \tag{12}$$

Dividing the equality (12) by $n>0$ and taking the limit for $n\to\infty$, we obtain the following equalities from the theorem 3.6 and definition 3.7, for all $x\in X$ and $y\in Y$,

$$I_{\mu,f}(\varphi^{-1}S,Q,y) = I_{\mu,f}(T,\varphi^{-1}Q,x) \tag{13}$$

and also

$$I_{\varphi o\mu,f}(S,Q,y)=I_{\mu_1,f}(S,Q,y)=I_{\mu,f}(T,\varphi^{-1}Q,x) \tag{14}$$

Hence, we have from the definition 3.10, for all $x\in X$ and $y\in Y$,

$$I_{\mu_1,f}(S,y)=\sup_{Q}\left\{I_{\mu_1,f}(S,Q,y)\,\middle|\,\begin{matrix}\text{is a finite fuzzy measurable partition}\\ \text{of } Y \text{ with } I_{\mu_1 f}(Q,y)<\infty\end{matrix}\right\}$$

by the equality (10)

$$=\sup_{\varphi^{-1}Q}\left\{I_{\mu_1,f}(T,\varphi^{-1}Q,x)\,\middle|\,\begin{matrix}\varphi^{-1}Q \text{ is a finite fuzzy measurable partition}\\ \text{of } X \text{ with } I_{\mu_1 f}(\varphi^{-1}Q,x)<\infty\end{matrix}\right\}$$

by the proposition 3.8 (iii)

$$=\sup_{P}\left\{I_{\mu_1,f}(T,P,x)\,\middle|\,\begin{matrix}P \text{ is a finite fuzzy measurable partition}\\ \text{of } X \text{ with } I_{\mu_1 f}(P,x)<\infty\end{matrix}\right\}.$$

Therefore the result follows from the Definition 3.10, for all $x\in X$ and $y\in Y$,

$$I_{\mu_1,f}(S,y)\le I_{\mu,f}(T,x)$$

$$=\sup_{Q}\left\{I_{\mu_1,f}(S,Q,y)\,\middle|\,\begin{matrix}\text{is a finite fuzzy measurable partition}\\ \text{of } Y \text{ with } I_{\mu_1 f}(Q,y)<\infty\end{matrix}\right\}$$

by the equality (10)

$$=\sup_{\varphi^{-1}Q}\left\{I_{\mu,f}(T,\varphi^{-1}Q,x)\,\middle|\,\begin{matrix}\varphi^{-1}Q \text{ is a finite fuzzy measurable partition}\\ \text{of } X \text{ with } I_{\mu f}(\varphi^{-1}Q,x)<\infty\end{matrix}\right\}$$

by the proposition 3.8 (iii)

$$\le\sup_{P}\left\{I_{\mu,f}(T,P,x)\,\middle|\,\begin{matrix}P \text{ is a finite fuzzy measurable partition}\\ \text{of } X \text{ with } I_{\mu f}(P,x)<\infty\end{matrix}\right\}.$$

Therefore the result follows from the definition 3.10, for all $x\in X$ and $y\in Y$,

$$I_{\mu_1,f}(S,y)\le I_{\mu,f}(T,x) \tag{15}$$

3.14 Definition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with

$I_{\mu,f}(P,x)<\infty$, for $x\in X$. Then,

if $\overset{+\infty}{\underset{i=-\infty}{V}} T^i P\equiv F$, then the fuzzy partition P is called a fuzzy generator of the σ-algebra F for T.

3.15 Corollary Let $(P_n)_{n\geq 1}$ be a family of the fuzzy measurable partitions such that

$P_1\subset P_2\subset\ldots\subset P_n\subset\ldots$ and $\overset{n}{\underset{i=1}{V}} P_i = X$.Then for all $x\in X$, Let $(P_n)_{n\geq 1}$ be a family of the fuzzy measurable partition such that $P_1\subset P_2\subset\ldots\subset P_n\subset\ldots$ and $\overset{n}{\underset{i=1}{V}} P_i = X$. Then for all $x\in X$, $I_{\mu,f}(T,x)=\lim_{n\to\infty} I_{\mu,f}(T,P_n,x)$

Proof We write the following equality from Definition 3.10, for all $x\in X$,

$$I_{\mu,f}(T,x)=\sup_{P_k}\left\{ I_{\mu,f}(T,P_k,x) \;\middle|\; \begin{array}{r} P_k \text{ is a finite fuzzy measurable partition} \\ \text{of } X \text{ with } I_{\mu f}(P_k,x)<\infty \end{array} \right\}$$

$$=\sup_{P_k}\left\{ \lim_{n\to\infty}\frac{1}{n} I_{\mu,f}\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P_k, x\right)\right\}$$

by the Theorem 3.6 and definition 3.10, we have the following the result for all $x\in X$,

$$I_{\mu,f}(T,x)=\lim_{k\to\infty} I_{\mu,f}(T,P_k,x) \tag{16}$$

3.16 Proposition Let P be a finite generating fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$, for $x\in X$. Then, for all $x\in X$, $I_{\mu,f}(T,x)=I_{\mu,f}(T,P,x)$.

Proof See, [1] and [7].

3.17 Lemma. $I_{\mu,f}(T,x)$ is an isomorphism invariant.

Proof Let (X,T) and (Y,S) be two fuzzy isomorphic dynamical systems with $\varphi: X\to Y$, $\varphi(T(x))=S(\varphi(x))$ for all $x\in X$. If P is a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$, for $x\in X$. Then, φP is a finite fuzzy measurable partition of the fuzzy dynamical system (Y,S) with $I_{\mu_1,f}(\varphi P,y)<\infty$, for $y\in Y$. Therefore for all $x\in X$ and $y\in Y$, by Theorem 3.6

$$I_{\mu_1,f}(S,\varphi P,y)=I_{\mu_1,f}(\varphi T\varphi^{-1},\varphi P,y)$$

$$=\lim_{n\to\infty}\frac{1}{n}I_{\mu_1,f}\left(\overset{n-1}{\underset{i=0}{V}}(\varphi T^{-i}\varphi^{-1})(\varphi P),y\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}I_{\mu_1,f}\left(\overset{n-1}{\underset{i=0}{V}}\varphi T^{-i}P,y\right) \tag{17}$$

by proposition 3.4 (v)

$$=\lim_{n\to\infty}\frac{1}{n}I_{\mu_1,f}\left(\varphi^{-1}(\overset{n-1}{\underset{i=0}{V}}\varphi T^{-i}\varphi P,\varphi^{-1}y\right)$$

since φ is surjective

$$=\lim_{n\to\infty}\frac{1}{n}I_{\mu_1,f}\left(\overset{n-1}{\underset{i=0}{V}}(T^{-i}P,x\right)$$

by theorem 3.6

$$=I_{\mu,f}(T,P,x).$$

3.18 Proposition Let (X,T) and (Y,S) be two fuzzy dynamical systems. Then for all $(x,y)\in X\times Y$, $I_{\mu\times\mu_1}(T\times S,x\times y)=I_{\mu,f}(T,x)+I_{\mu_1,f}(S,y)$, where $T\times S$ is a fuzzy transformation defined on the fuzzy product space $(X\times Y,T\times S)$ with $(T\times S)\times(x\times y)=(Tx,Sy)$.

Proof Let $(P_n)_{n\geq1}$ (resp. $(P_n')_{n\geq1}$) be an increasing fuzzy sequence of the fuzzy partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P_n,x)<\infty$ for all $n\in N$, (resp. fuzzy dynamical system (Y,S) with $I_{\mu,f}(P_n',x)<\infty$ for all $n\in N$, $y\in Y$) which generates F (resp. F_1). Each P_n induces a fuzzy partition Q_n of the fuzzy product space $(X\times Y,T\times S)$. The elements of Q_n being of the form $F\times Y$, where F runs through the elements of P_n. Similarly P_n' induces a partition Q_n' of the fuzzy product space $(X\times Y,T\times S)$. It is easy $U_n=Q_n\times Q_n'$ is an increasing fuzzy sequence of the fuzzy product space $x\in X$ $(X\times Y,T\times S)$ which generates $F\times F_1$ since Q_n and Q_n' are independent, one has for all $(x,y)\in X\times Y$;

$$I_{\mu\times\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}U_n, x\times y\right)=I_{\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q_n, x\times y\right)+I_{\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q'_n, x\times y\right) \quad (18)$$

But clearly;

$$I_{\mu\times\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q_n, x\times y\right)=I_{\mu,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P_n, x\right) \quad (19)$$

and

$$I_{\mu\times\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q'_n, x\times y\right)=I_{\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P'_n, y\right) \quad (20)$$

Thus we obtain for all $(x,y)\in X\times Y$;

$$I_{\mu\times\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}U_n, x\times y\right)=I_{\mu,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P_n, x\right)+I_{\mu_1,f}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P'_n, y\right) \quad (21)$$

Therefore, dividing the last equality (21) by $k>0$ and taking the limit for $k\to\infty$, one obtains the following equality from the theorem 3.6, for all $(x,y)\in X\times Y$;

$$I_{\mu\times\mu_1,f}\left(T\times S, U_n, x\times y\right)=I_{\mu,f}\left(T,P_n,x\right)+I_{\mu_1,f}\left(S,P'_n,y\right) \quad (22)$$

Taking the limit for $k\to\infty$, one has therefore the result from the corollary 3.15, for all $(x,y)\in X\times Y$;

$$I_{\mu\times\mu_1,f}\left(T\times S, x\times y\right)=I_{\mu,f}\left(T,x\right)+I_{\mu_1,f}\left(S,y\right) \quad (23)$$

4. Fuzzy local information function

4.1 Definition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$ for each $x\in X$.

The quantity $\mathrm{l}_{\mu,f}(T,x)=I_{\mu,f}(T,x)-I_{\mu,f}(T,P,x)$ for each $x\in X$ is called a fuzzy local information function.

4.2 Proposition *Let* P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$ for each $x\in X$.

i) $\mathrm{l}_{\mu,f}(T,x)\geq 0$.

ii) $l_{\mu,f}(Id,x)=0$.

iii) If P is a finite fuzzy partition which generates F, then $l_{\mu,f}(T,x)=0$.

Proof (i) and (ii) are trivial from the proposition 3.3 (i) and (ii), corollary 3.10 (i) and (ii) and definition 4.1.

iii) Suppose that P is a generating fuzzy partition of (X,T) with $I_{\mu,f}(P,x)<\infty$ for each $x\in X$. Write the following equality from the definition 4.1, for each $x\in X$;

$$l_{\mu,f}(T,x)=I_{\mu,f}(T,x)-I_{\mu,f}(T,P,x) \tag{24}$$

Since

$$I_{\mu,f}(T,x)=I_{\mu,f}(T,P,x) \tag{25}$$

for each $x\in X$ from the proposition 3.16, we obtain thus the result

$$l_{\mu,f}(T,x)=0 \tag{26}$$

4.3 Proposition Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$ for each $x\in X$.

i) For $k>0$, $l_{\mu,f}(T^k,x)=kl_{\mu,f}(T,x)$

ii) If T is an invertible fuzzy transformation and $k\in Z$, then $l_{\mu,f}(T^k,x)=|k|l_{\mu,f}(T,x)$

Proof i) Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x)<\infty$ for each $x\in X$.

Since for each $x\in X$,

$$I_{\mu,f}(T^k,P,x)=kI_{\mu,f}(T,P,x) \tag{27}$$

for $k>0$ from the proposition 3.9 (i) and for each $x\in X$,

$$I_{\mu,f}(T^k,x)=kI_{\mu,f}(T,x) \tag{28}$$

for $k>0$ from the proposition 3.11.(iii) we write thus the following equality for each $x\in X$;

$$I_{\mu,f}(T^k,x)-I_{\mu,f}(T^k,P,x)=kI_{\mu,f}(T,x)-kI_{\mu,f}(T,P,x) \tag{29}$$

Hence we have the result from the definition 4.1 for each $x \in X$ and $k > 0$;

$$1_{\mu,f}(T^k, x) = k 1_{\mu,f}(T, x) \tag{30}$$

ii) Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$ for each $x \in X$. As T is an invertible fuzzy transformation and $k \in Z$, we can also write the following equalities by proposition 3.9.(ii) and proposition 3.11.(iv), for each $x \in X$;

$$I_{\mu,f}(T^k, P, x) = |k| I_{\mu,f}(T, P, x) \tag{31}$$

and

$$I_{\mu,f}(T^k, x) = |k| I_{\mu,f}(T, x) \tag{32}$$

Therefore, we have for each $x \in X$;

$$I_{\mu,f}(T^k, x) - I_{\mu,f}(T^k, P, x) = |k| I_{\mu,f}(T, x) - |k| I_{\mu,f}(T, P, x) \tag{33}$$

Thus, the result follows from the definition 4.1. for each $x \in X$;

$$1_{\mu,f}(T^k, x) = |k| 1_{\mu,f}(T, x) \tag{34}$$

4.4 Proposition Let P and Q be two finite fuzzy measurable partitions of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$ for each $x \in X$ and (Y,S) with $I_{\mu_1,f}(S,y) < \infty$ for each $y \in Y$. If the fuzzy dynamical system (Y,S) is a fuzzy factor of (X,T), then, for each $x \in X$ and $y \in Y$;

$$1_{\mu_1,f}(S,y) \le 1_{\mu,f}(T,x) + I_{\mu,f}(T,P,x) - I_{\mu_1,f}(S,Q,y).$$

Proof We write the following inequality from the Proposition 3.13 for each $x \in X$ and $y \in Y$

$$I_{\mu_1,f}(S,y) \le I_{\mu,f}(T,x) \tag{35}$$

Let Q be a finite fuzzy measurable partition of the fuzzy factor (Y,S) with $I_{\mu_1,f}(S,y) < \infty$ for each $y \in Y$. As $I_{\mu_1,f}(S,Q,y) \ge 0$ from the Proposition 3.8 (**i**), one has the following inequality, for each $x \in X$ and $y \in Y$;

$$I_{\mu_1,f}(S,y) - I_{\mu_1,f}(S,Q,y) \le I_{\mu,f}(T,x) - I_{\mu_1,f}(S,Q,y) \tag{36}$$

Therefore, one obtains from the definition 4.1 each $x \in X$ and $y \in Y$

$$1_{\mu_1,f}(S,y) \le I_{\mu,f}(T,x) - I_{\mu_1,f}(S,Q,y) \tag{37}$$

Let P be a finite fuzzy measurable partition of the fuzzy dynamical system (X,T) with $I_{\mu,f}(P,x) < \infty$ for each $x \in X$.

As $I_{\mu,f}(T,P,x) \ge 0$ from the proposition 3.8 (i), we can also write the following inequality, for each $x \in X$ and $y \in Y$;

$$1_{\mu_1,f}(S,y) - I_{\mu,f}(T,P,x) \le I_{\mu,f}(T,x) - I_{\mu,f}(T,P,x) - I_{\mu_1,f}(S,Q,y) \tag{38}$$

Hence, we obtain the result from the definition 4.1, for each $x \in X$ and $y \in Y$;

$$1_{\mu_1,f}(S,y) \le 1_{\mu,f}(T,x) + I_{\mu,f}(T,P,x) - I_{\mu_1,f}(S,Q,y) \tag{39}$$

4.5 Proposition Let (X,T) and (Y,S) be two fuzzy dynamical systems.

Then, for all $(x,y) \in X \times Y$, $1_{\mu\times\mu_1,f}(T\times S, x\times y) = 1_{\mu,f}(T,x) + 1_{\mu_1,f}(S,y)$.

Where $T \times S$ is a fuzzy transformation defined on the fuzzy product space $X \times Y$.

Proof Let P and Q be two finite fuzzy measurable partitions of X and Y respectively with $I_{\mu,f}(P,x) < \infty$ for each $x \in X$ and $I_{\mu_1,f}(S,y) < \infty$ for each $y \in Y$.

Then, we can write the following equalities by proposition 3.15, for each $x \in X$ and $y \in Y$;

$$I_{\mu\times\mu_1,f}(T\times S, P\times Q, x\times y) = I_{\mu,f}(T,P,x) + I_{\mu_1,f}(S,Q,y) \tag{40}$$

and

$$I_{\mu\times\mu_1,f}(T\times S, x\times y) = I_{\mu,f}(T,x) + I_{\mu_1,f}(S,y) \tag{41}$$

Therefore we have, for all $(x,y) \in X \times Y$,

$$\begin{aligned} I_{\mu\times\mu_1,f}(T\times S, x\times y) - I_{\mu\times\mu_1,f}(T\times S, P\times Q, x\times y) = \\ I_{\mu,f}(T,x) - I_{\mu,f}(T,P,x) + I_{\mu_1,f}(S,y) - I_{\mu_1,f}(S,Q,y) \end{aligned} \tag{42}$$

Hence, the result follows from the Definition 4.1, for all $(x,y) \in X \times Y$;

$$1_{\mu\times\mu_1,f}(T\times S, x\times y) = 1_{\mu,f}(T,x) + 1_{\mu_1,f}(S,y) \tag{43}$$

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