BALKANJM 02 (2014) 141-149

Contents lists available at BALKANJM

BALKAN JOURNAL OF MATHEMATICS

journal homepage: www.balkanjm.com

Hamilton Equations on Three-Dimensional Space of Mechanical Systems

Zeki Kasap[*a]

[a] *Department of Mathematics, Faculty of Science, Institute of Science, University of Çanakkale Onsekizmart, Çanakkale-Turkey*.

ARTICLE INFO

Article history:
Received 27.10.2013
Accepted 12.02.2014
Available online 30.05.2014

Keywords:
Kähler-Einstein Manifold
Hamiltonian
Mechanical System

ABSTRACT

The paper deal with the Hamiltonian formalism of mechanical systems using on three-dimensional space which represent an interesting multidisciplinary field of research. In this study, the motion route of bodies in space mathematically will be modeled. We, as a result modeling obtained of partial differential equations, will be solved by symbolic computational program. Also, the geometrical-physical results related to on three-dimensional space of mechanical systems.

1. Introduction

There are lots of applications on differential geometry and mathematical physics that their are used in many areas. One of the most important applications of differential geometry is on geodesics. A geodesic is the shortest route between given two points. Geodesics can be found with the help of the Hamilton equations. We can say that differential geometry provides a suitable field for studying Hamiltonians of classical mechanics, analytic mechanics and field theory. The dynamic equations for moving bodies are obtained according to Hamiltonian mechanics formulation by many authors and are illustrated as follows. There have been many studies about Hamiltonian dynamics, mechanics, formalisms, systems and equations. There are real, complex, paracomplex and other analogues. As is known, it is possible to produce different analogous in different spaces. Now, we give some examples as follows: *Liu* showed that if a Hamiltonian function and the initial state of the atoms in the system are known, one can compute the instantaneous positions and momenta of the atoms at all successive times [1]. *Bradley* demonstrated that it is possible to write Hamiltonians as in Newton's second rule $F = ma$ that lets you avoid having to deal with vector-valued force balances. They not only

[*] Corresponding author: *E-mail*: zekikasap@hotmail.com (Z. Kasap).

make hideous mechanics problems simple, but they also expose deep symmetries and conserved properties. Rigid-body dynamics is the study of the movement of the objects like baseballs, planets, tops, and snowflakes through space. Gravitation concerns the intricacies of the n-body problem: n masses pulling on one another in the standard $G\frac{mM}{r^2}$ way [2]. *Ioffe* explored that dynamic optimization problems for systems governed by differential inclusions are considered. The main focus is on the structure of and interrelations between necessary optimally conditions stated in terms of Hamiltonian formalisms [3]. *Antoniou* and *Pronko* suggested the Hamiltonian approach for fluid mechanics based on the dynamics formulated in terms of Lagrangian variables [4]. *Antoniou* and *Pronko* proposed a Hamiltonian approach to .uid mechanics based on the dynamics formulated in terms of Lagrangian variables. They also discussed the difference between the Eulerian and the Lagrangian descriptions, pointing out the incompleteness of the former. The constructed formalism was also applicable to an ideal plasma. They concluded with several remarks about quantizing the fluid [5]. The gravitational two-body problem in given was generalized by *Barker* and *O'Connell* [6]. *Becker* and *Scherpen* examined that a Lagrangian had been developed for leading the equations of motion which are isomorphic to the full Navier-Stokes equation, including dissipation [7]. *Spotti* investigated how Fano manifolds equipped with a Kähler-Einstein metric can degenerate as metric spaces (in the Gromov-Hausdorff topology) and some of the relations of this question with Algebraic Geometry [8]. *Tekkoyun* showed that paracomplex analogue of the Euler-Lagrange equations was obtained in the framework of para-Kählerian manifold and the geometric results on a paracomplex mechanical systems were found [9]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by *Tekkoyun* and *Sari* [10]. *Tekkoyun* and *Yayli* presented generalized-quaternionic Kählerian analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic Kählerian mechanical systems are provided [11]. *Kasap* and *Tekkoyun* introduced Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research. Also, the geometrical, relativistical, mechanical and physical results related to para/pseudo-Kähler mechanical systems were given, too [12]. *Kasap* demonstrated Weyl-Euler-Lagrange and Weyl-Hamilton equations on R_n^{2n} which is a model of tangent manifolds of constant W-Sectional curvature [13].

From the above, we show that some examples of the Hamiltonian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, rotating pendulum, molecular and fluid dynamics, LC networks, Atwood's machine, symmetric top etc. In the present paper, we provide equations related to Hamiltonian mechanical systems on three-dimensional space. Also, we will be present on three-dimensional space, its results and solutions.

2. Preliminaries

In this study, all manifolds and geometric structures are supposed that differentiable. The Einstein summation convention $(\sum a_j x^j = a_j x^j)$ is in use. Also, TM is tangent manifold, T^*M is cotangent manifold of a manifold M and M is an n-dimensional differentiable manifold. Additionally, vector fields, the set of para-complex numbers, the set of para-complex functions on TM, the set of para-complex vector fields on TM and the set of para-complex 1-forms on TM are represented by $\{X,Y\}, A$, $F(TM)$, $\chi(TM)$ and $\wedge^1(TM)$, respectively.

3. J-Holomorphic Curves

A pseudoholomorphic curve (or J-holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy-Riemann equation. Introduced in 1985 by *Gromov*, pseudoholomorphic curves have since revolutionized the study of symplectic manifolds. The theory of J-holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry, making it possible to study the global structure of symplectic manifolds. The methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic. The effect of this is to ensure that one is looking at persistent rather than accidental features of these curves. However, the perturbed structure may no longer be integrable, and so again one is led to the study of curves which are holomorphic with respect to some non-integrable almost complex structure J. A complex-valued function f of a complex variable z is said to be holomorphic at a point a if it is differentiable at every point within some open disk centered at a. Pseudosphere is negative curvature [14].

4. The Cauchy.Riemann Equation

The Cauchy-Riemann differential equations in complex analysis consist of a system of two partial differential equations which must be satisfied if it is know that a complex function is complex differentiable. Moreover, the equations are necessary and sufficient conditions for complex differentiation once it seen that its real and imaginary parts are differentiable real functions of two variables. The Cauchy-Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \tag{1}$$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z=x+iy$, $f(x+iy)=u(x,y)+iv(x,y)$.

5. Symplectic Geometry

A symplectic manifold (M,ω) is a smooth manifold (M) equipped with a closed nondegenerate differential 2-form (ω) called the symplectic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds, e.g., in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field. The set of all possible

configurations of a system is modelled as a manifold, and this manifold's cotangent bundle describes the phase space of the system. The basic example of an almost complex symplectic manifold is standard Euclidean space $(\mathrm{R}^{2n}, \omega_0)$ with its standard almost complex structure J_0 obtained from the usual identification with C^n.
Thus, one sets

$$z_j = x_{2j-i} + i\, x_{2j} \tag{2}$$

for $j = 1,...,n$ and defines J_0 by

$$J_0(\partial_{2j-1}) = \partial_{2j}, \quad J_0(\partial_{2j}) = -\partial_{2j-1} \tag{3}$$

where $\partial_j = \partial / \partial x_j$ is the standard basis of $T_x R^{2n}$ [14].

6. Almost (para)-Complex Structure and Manifolds

Definition 1: Let M be a smooth manifold of real dimension $2n$. We say that a smooth atlas A of M is holomorphic if for any two coordinate charts $z : U \to U' \subset C^m$ and $\omega : V \to V' \subset C^m$ in A, the coordinate transition map $z \circ \omega^{-1}$ is holomorphic. Any holomorphic atlas uniquely determines a maximal holomorphic atlas, and a maximal holomorphic atlas is called a complex structure for M. We say that M is a complex manifold of complex dimension n if M comes equipped with a holomorphic atlas. Any coordinate chart of the corresponding complex structure will be called a holomorphic coordinate chart of M. A Riemann surface or complex curve is a complex manifold of complex dimension 1.

Definition 2: Let M be a differentiable manifold of dimension $2n$ and suppose J is a differentiable vector bundle isomorphism $J : TM \to TM$ such that $J_x : T_x M \to T_x M$ is a complex structure for $T_x M$, i.e. $J^2 = -I$ where I is the identity vector bundle isomorphism and $J^2 = J \circ J$. Then J is called an almost-complex structure for the differentiable manifold M. A manifold with a fixed almost-complex structure is called an almost-complex manifold.

Definition 3: Let be V a vector space over R. Recall that a paracomplex structure on V is a linear operator J on V such that $J^2 = I$, and I is the identity operator on V. A prototypical example of a paracomplex structure is given by the map $J : V \to V$, where $V = R^n \oplus R^n$. An almost-paracomplex structure on M a manifold is a differentiable map $J : TM \to TM$ on the tangent bundle TM of M such that J preserves each fiber. A manifold with a fixed almost paracomplex structure is called an almost paracomplex manifold. A celebrated theorem of Newlander and Nirenberg [15] says that an almost (para) complex structure is a (para) complex structure if and only if its Nijenhuis tensor or torsion vanishes.

Theorem 1: The almost (para)-complex structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J[X,Y]=[JX,JY]-J[X,JY]-J[JX,Y]-[X,Y]. \tag{4}$$

The tensor $(2,1)$ is called the Nijenhuis tensor (4). We say that J is torsion free if $N_J=0$. (Para)-Complex Nijenhuis tensor of an almost (para)-complex manifold (M,J) is given by (4). It disappears if and only if J is an integrable almost (para)-complex structure, i.e. given any point $P\in N$, there are local coordinates which are centered at P so

$$J^*(dx)=dx+dz,\ J^*(dy)=dy+dz,\ J^*(dz)=-dz. \tag{5}$$

The structure J^* is the dual form of the structure J. These structures holomorphic property are

$$\begin{aligned} J^{*2}(dx)&=J^*\circ J^*(dx)=J^*(dx+dz)=J^*(dx)+J^*(dz)=dx+dz-dz=dx,\\ J^{*2}(dy)&=J^*\circ J^*(dy)=J^*(dy+dz)=J^*(dy)+J^*(dz)=dy+dz-dz=dy,\\ J^{*2}(dz)&=J^*\circ J^*(dz)=J^*(-dz)=-J^*(dz)=dz. \end{aligned} \tag{6}$$

As can be seen from above $J^{*2}=I$ are paracomplex structures (6). The system are based on three variables and three-dimensional for (x,y,z). In this study, above holomorpfic structures will be use.

7. Hamiltonian System

Definition 4: [16,17,18]: Let M is the base manifold of dimension n and its cotangent manifold T^*M. By a symplectic form we mean a $2-$form Φ on T^*M. Let (T^*M,Φ) be a symplectic manifold, there is a unique vector field X_H on T^*M and $H:T^*M\to R$ is called as Hamiltonian function. Where $H=T+V$ and T is kinetic energy and V is potential energy such that Hamiltonian dynamical equation is determined by

$$i_{X_H}=dH. \tag{7}$$

We say X_H is locally Hamiltonian vector field. Φ is closed and also shows the canonical symplectic form so that $\Phi=-d\Omega$, $\Omega=J^*(\omega)$, J^* a dual of J, ω a $1-$form on T^*M. The triple (T^*M,Φ,X_H) is named Hamiltonian system which is defined on the cotangent bundle T^*M. From the local expression for X_H we see that $(q^i(t),p_i(t))$ is an integral curve of X_H if Hamilton's equations is expressed as follows:

$$q^i=\frac{\partial H}{\partial p_i},\ \dot{p}_i=-\frac{\partial H}{\partial q^i}. \tag{8}$$

8. Hamilton Equations on Three-Dimensional Space

Now, we will present Hamilton equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on three-dimensional space. Let (M, J^*) be on three-dimensional space. Suppose that the complex structures, a Liouville form and a $1-$form on three-dimensional space are shown by J^*, Ω and ω respectively. Consider a $1-$form ω such that

$$\omega = -ydx - xdy + zdz \tag{9}$$

Then, we obtain the Liouville form as follows:

$$\Omega = J^*(\omega) = -y(dx+dz) - x(dy+dz) - zdz \tag{10}$$

It is well known that if Φ is a closed on three-dimensional space, then Φ is also a symplectic structure on (M, J^*). Therefore the $2-$form $\Phi = -d\Omega$ indicates the canonical symplectic form and derived from the $1-$form to find to mechanical equations. Then the $2-$form Φ is calculated as below:

$$\Phi = \begin{bmatrix} -\frac{\partial y}{\partial x}dx\wedge dx - -\frac{\partial y}{\partial x}dx\wedge dz - \frac{\partial x}{\partial x}dx\wedge dy - \frac{\partial x}{\partial x}dx\wedge dz - \frac{\partial z}{\partial x}dx\wedge dz \\ -\frac{\partial y}{\partial y}dy\wedge dx - -\frac{\partial y}{\partial y}dy\wedge dz - \frac{\partial x}{\partial y}dy\wedge dy - \frac{\partial x}{\partial y}dy\wedge dz - \frac{\partial z}{\partial y}dy\wedge dz \\ -\frac{\partial y}{\partial z}dz\wedge dx - -\frac{\partial y}{\partial z}dz\wedge dz - \frac{\partial x}{\partial z}dz\wedge dy - \frac{\partial x}{\partial z}dz\wedge dz - \frac{\partial z}{\partial z}dz\wedge dz \end{bmatrix}. \tag{11}$$

Take a vector field X_H so that called to be Hamiltonian vector field associated with Hamiltonian energy H and determined by

$$X_H = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}. \tag{12}$$

$\Phi(X_H)$ will be calculated using Φ and X_H. So,

$$\Phi(X_H) = -d\Omega = [2dy\wedge dx - dz\wedge dx - dz\wedge dy](X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}). \tag{13}$$

Calculations use external product feature. These properties are

$$\begin{aligned} f\wedge g &= -g\wedge f, \\ f\wedge g(v) &= f(v)g - g(v)f. \end{aligned} \tag{14}$$

We have

$$
\begin{aligned}
i_{X_H}\Phi &= \Phi(X_H) \\
&= 2.X\left[dy(\frac{\partial}{\partial x})dx - dx(\frac{\partial}{\partial x})dy\right] - X\left[dz(\frac{\partial}{\partial x})dx - dx(\frac{\partial}{\partial x})dz\right] - X\left[dz(\frac{\partial}{\partial x})dy - dy(\frac{\partial}{\partial x})dz\right] \\
&+ 2.Y\left[dy(\frac{\partial}{\partial y})dx - dx(\frac{\partial}{\partial y})dy\right] - Y\left[dz(\frac{\partial}{\partial y})dx - dx(\frac{\partial}{\partial y})dz\right] - Y\left[dz(\frac{\partial}{\partial y})dy - dy(\frac{\partial}{\partial y})dz\right] \\
&+ 2.Z\left[dy(\frac{\partial}{\partial z})dx - dx(\frac{\partial}{\partial z})dy\right] - Z\left[dz(\frac{\partial}{\partial z})dx - dx(\frac{\partial}{\partial z})dz\right] - Z\left[dz(\frac{\partial}{\partial z})dy - dy(\frac{\partial}{\partial z})dz\right]
\end{aligned} \tag{15}
$$

Furthermore, the differential of Hamiltonian energy H is obtained by

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz. \tag{16}$$

From $i_{X_H}\Phi = dH$ the Hamiltonian vector field is found as follows:

$$2.\mathrm{X}dy + \mathrm{X}dz + 2.\mathrm{Y}\,dx + \mathrm{Y}\,dz - \mathrm{Z}dx = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz \tag{17}$$

and

$$X = \frac{-1}{2}\frac{\partial H}{\partial y},\ Y = \frac{\partial H}{\partial z} + \frac{1}{2}\frac{\partial H}{\partial y},\ Z = 2\frac{\partial H}{\partial z} + \frac{\partial H}{\partial y} - \frac{\partial H}{\partial x} \tag{18}$$

and then

$$X_H = \left(\frac{-1}{2}\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial x} + \left(\frac{\partial H}{\partial z} + \frac{1}{2}\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial y} + \left(2\frac{\partial H}{\partial z} + \frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\right)\frac{\partial}{\partial z} \tag{19}$$

Consider the curve and its velocity vector

$$
\begin{aligned}
&\alpha : I \subset R \to M,\ \ \alpha(t) = \big(x(t), y(t), z(t)\big) \\
&\dot{\alpha}(t) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}
\end{aligned} \tag{20}
$$

such that an integral curve of the Hamiltonian vector field Z_H, i.e.,

$$X_H\big(\alpha(t)\big) = \frac{\partial \alpha}{\partial t},\ t \in I. \tag{21}$$

Then, we find the following equations;

$$
\begin{aligned}
&I. && \frac{dx}{dt} = \frac{-1}{2}\frac{\partial H}{\partial y} \\
&II. && \frac{dy}{dt} = \frac{\partial H}{\partial z} + \frac{1}{2}\frac{\partial H}{\partial y} \\
&III. && \frac{dz}{dt} = 2\frac{\partial H}{\partial z} + \frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}.
\end{aligned} \tag{22}
$$

Hence, the equations introduced in (22) are named Hamilton equations on three-dimensional space and then the triple (M, X_H) is said to be a Hamiltonian mechanical system on three-dimensional space.

9. Equations Solution

These found (22) are partial differential equation depending on the time and there are on three-dimensional space. We can solve these equations using the symbolic computational program. The software codes of these equations,

Equations Codes

I. $\mathrm{dif1 := diff(x(t), t) = 0.5 * diff(H_1(x, y, z, t), y)}$,
II. $\mathrm{dif2 := diff(y(t), t) = diff(H_2(x, y, z, t), z) + 0.5 * diff(H_2(x, y, z, t), y)}$,
III. $\mathrm{dif3 := diff(z(t), t) = 2 * diff(H_3(x, y, z, t), z) + diff(H_3(x, y, z, t), y) - diff(H_3(x, y, z, t), x)}$,

(23)

For example at (23), we choose as special case of $x(t)$, $y(t)$, $z(t)$ and they solutions as follows:

Closed Solutions of Equations

I. $\mathrm{H_1(x, y, z, t) = -2 * y - 2 * \cos(t) * y + F_1(x, z, t)}$; for $x(t) := t + \sin(t)$,
II. $\mathrm{H_2(x, y, z, t) = 1/t * z + F_2(x, y - 1/2 * z, t)}$; for $y(t) := \ln(t)$,
III. $\mathrm{H_3(x, y, z, t) = -2 * t * x + F_3(y + x, z + 2 * x, t)}$; for $z(t) := t^2$.

(24)

10. Conclusion

In this study, the paths of Hamiltonian vector fields for X_H on three-dimensional space are the solutions Hamilton equations raised in (22) on three-dimensional space for mechanical systems. Also we found the closed solutions of partial differential equations (24) that they are the equations of motion of objects in space. Nowadays, well-known Hamiltonian models have emerged as a very important tool since they present a simple method to describe the model for mechanical systems. Furthermore, the metrics are interpreted as the gravitational potential, as in general relativity, and the corresponding forms are interpreted as the electromagnetic otentials.

Furthermore, the equations found by (22) easily seen extremely useful in applications from Hamiltonian Mechanics, Quantum Physics, Optimal Control, Biology and Fluid Dynamics. For this reason, the found equations are only considered to be a first step to realize how a generalized on three-dimensional space geometry.

They has been used in solving problems in different physical area. Our proposal for future research, the Hamiltonian mechanical equations derived on a generalized on three-dimensional space are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics [19,20,21].

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