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Euler-Lagrange Equations of Moving Objects on Flat Manifolds

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ABSTRACT

This article relates to the equations of moving objects in space. Hence Lagrangian formalism of mechanical systems on Flat manifolds that represent an interesting multidisciplinary field of research. We, as a result modeling obtained of partial differential equations, have be solved by the symbolic computational program. Also, the geometrical-physical results related to on Flat manifolds of mechanical systems.

1. Introduction

Mathematical physics and differential geometry has lots of different applications for our life that these applications are used in many areas. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained for Lagrangian mechanics. There are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations. There are real, complex, paracomplex and other analogues for these studies. It is well-known that Euler-Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows. Some examples of the Euler-Lagrangian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendullum, molecular and fluid dynamics, LC networks, Atwood's machine, symmetric top etc. Lets remember some work done. *Kasap* examined Weyl-Euler-Lagrange and Weyl-Hamilton equations on R_n^{2n}. Additionally, he was used a model of tangent manifolds of constant $W-$sectional curvature [1]. *Kapovich* proved an existence theorem for flat conformal structures on finite-sheeted coverings over a wide class of Haken manifolds [2]. *Schwartz* considered asymptotically at Riemannian manifolds with

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nonnegative scalar curvature [3]. *Kulkarni* explained some new examples of conformally flat manifolds, as a step toward a classification of such manifolds up to conformal equivalence [4]. *Dotti* and *Miatello* purposed the real cohomology ring of low dimensional compact flat manifolds endowed with one of these special structures [5]. *Szczepanski* gave a list of six dimensional at Kähler manifolds. Moreover, we present an example of eight dimensional at Kähler manifold M with finite $Out(\pi_1(M))$ group [6]. *Ge*, *Wang* and *Wu* showed that the mass of an asymptotically flat n-manifold is a geometric invariant [7]. *Gonzalez* look at complete, locally conformally flat (lcf) metrics defined on a domain $\Omega \subset S^n$ [8]. *Akbulut* and *Kalafat* constructed infinite families of non-simply connected locally conformally flat (LCF) 4-manifolds realizing rich topological types [9].

2. Preliminaries

In this study, all manifolds and geometric structures are supposed that differentiable. The Einstein summation convention ($\sum a_j x^j = a_j x^j$) is in use. Also, TM is tangent manifold, of a manifold M and M is an n -dimensional differentiable manifold. Additionally, vector fields, the set of paracomplex functions on TM, the set of paracomplex vector fields on TM and the set of paracomplex $1-$forms on TM are represented by $\{X,Y\}$, $F(M)$, $\chi(TM)$ and $\wedge^1(TM)$, respectively.

3. J -Holomorphic Curves

A pseudoholomorphic curve ($J-$holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold such that satisfies the Cauchy-Riemann equation. Introduced in 1985 by *Gromov*, pseudoholomorphic curves have since revolutionized the study of symplectic manifolds. The theory of $J-$holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry, making it possible to study the global structure of symplectic manifolds. The methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic. The effect of this is to ensure that one is looking at persistent rather than accidental features of these curves. Nevertheless, the perturbed structure may no longer be integrable, and so again one is led to the study of curves which are holomorphic with respect to some non-integrable almost complex structure J. f is a complex-valued function of a complex variable z such that it said to be holomorphic at a point a if it is differentiable at every point within some open disk centered at a. Also, pseudosphere is called negative curvature [10].

4. Almost (para)-Complex and Tangent Structure

Definition 1: Let M be a differentiable manifold of dimension $2n$ and suppose J is a differentiable vector bundle isomorphism $J: TM \to TM$ so that $J_x : T_x M \to T_x M$ is a complex structure for $T_x M$, i.e. $J^2 = -I$ where I is the identity vector bundle isomorphism and $J^2 = J \circ J$. Then J is called an almost-complex structure for the differentiable manifold M. A manifold with a fixed almost-complex structure is called an almost-complex manifold.

Definition 2: Let be V a vector space over ℝ. Recall that a paracomplex structure on V is a linear operator J on V such that $J^2 = I$, and I is the identity operator on V. A prototypical example of a paracomplex structure is given by the map $J: V \to V$, where $V = R^n \oplus R^n$. An almost-paracomplex structure on M a manifold is a differentiable map $J: TM \to TM$ on the tangent bundle TM of M such that J preserves each fiber. A manifold with a fixed almost paracomplex structure is called an almost paracomplex manifold.

Definition 3: Let be V a vector space over ℝ. Recall that a tangent (exact) structure on V is a linear operator J on V such that $J^2 = 0$, where $J^2 = J \circ J$, and I is the identity operator on V. A celebrated theorem of Newlander and Nirenberg [11] says that an almost (para) complex structure is a (para)complex structure if and only if its Nijenhuis tensor or torsion vanishes.

Definition 4: Let suppose that X is a vector field. A vector-valued function with cartesian coordinates $(X_1,...,X_n)$ and $\alpha(t)$ a parametric curve with cartesian coordinates $(\alpha_1(t),...,\alpha_n(t))$. Then $\alpha(t)$ is an integral curve of X if it is a solution of the following autonomous system of ordinary differential equations: $\frac{d\alpha_1}{dt} = X_1(\alpha_1,...,\alpha_n)$,…, $\frac{d\alpha_n}{dt} = X_n(\alpha_1,...,\alpha_n)$. Such a system may be written as a single vector equation $X(\alpha(t)) = \alpha'(t) = \frac{d\alpha}{dt}$, and so the curve $\alpha(t)$ is tangent at each point to the vector field X.

Theorem 1: J is the almost complex structure on M such that it integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]. \tag{1}$$

The tensor (2,1) is called the Nijenhuis tensor (1). We say that J is torsion free if $N_J = 0$. Paracomplex Nijenhuis tensor of an almost (para)-complex manifold (M,J) is given by (1). Let $(x_1,...,x_{2n})$ be a local coordinate system. The torsion tensor is bilinear, for if $X = \frac{\partial}{\partial x_j}$ and $Y = \frac{\partial}{\partial x_k}$ are vector fields and J_j^i are the components of J, then by direct calculation the i^{th} component of the torsion tensor is given by

$$N\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)^i = N_{jk}^i = \sum_{h=0}^{2n}\left(J_j^h \partial_h J_k^i - J_k^h \partial_h J_j^i - J_h^i \partial_j J_k^h - J_h^i \partial_k J_j^h\right), \tag{2}$$

where ∂_h denotes partial differentiation ∂_{x_h}. It disappears if and only if J is an integrable almost (para)-complex structure, i.e. given any point $P \in N$, there are local coordinates which are centered at P so,

$$(1).\ J\left(\frac{\partial}{\partial x_1}\right)=\cos(x_3)\frac{\partial}{\partial y_1}+\sin(x_3)\frac{\partial}{\partial y_2},\quad (2).\ J\left(\frac{\partial}{\partial x_2}\right)=-\sin(x_3)\frac{\partial}{\partial y_1}+\cos(x_3)\frac{\partial}{\partial y_2},\ (3).\ J\left(\frac{\partial}{\partial x_3}\right)=\frac{\partial}{\partial y_3},$$
$$(4).\ J\left(\frac{\partial}{\partial y_1}\right)=-\cos(x_3)\frac{\partial}{\partial x_1}+\sin(x_3)\frac{\partial}{\partial x_2},\ (5).\ J\left(\frac{\partial}{\partial y_2}\right)=-\sin(x_3)\frac{\partial}{\partial x_1}-\cos(x_3)\frac{\partial}{\partial x_2},\ (6).\ J\left(\frac{\partial}{\partial y_3}\right)=-\frac{\partial}{\partial x_3}. \tag{3}$$

The above structures (3) have been taken from [12]. These structures holomorphic property are as follows.

$$\begin{aligned}(1)\ J^2\left(\frac{\partial}{\partial x_1}\right)&=J\circ J\left(\frac{\partial}{\partial x_1}\right)=J\left(\cos(x_3)\frac{\partial}{\partial y_1}+\sin(x_3)\frac{\partial}{\partial y_2}\right)=\cos(x_3)J\left(\frac{\partial}{\partial y_1}\right)+\sin(x_3)J\left(\frac{\partial}{\partial y_2}\right)\\&=\cos(x_3)\left[-\cos(x_3)\frac{\partial}{\partial x_1}+\sin(x_3)\frac{\partial}{\partial x_2}\right]+\sin(x_3)\left[-\sin(x_3)\frac{\partial}{\partial x_1}-\cos(x_3)\frac{\partial}{\partial x_2}\right]\\&=\left[-\cos^2(x_3)-\sin^2(x_3)\right]\frac{\partial}{\partial x_1}=-\frac{\partial}{\partial x_1}.\end{aligned} \tag{4}$$

Similar to this process include the following.

$$(2)\ J^2\left(\frac{\partial}{\partial x_2}\right)=-\frac{\partial}{\partial x_2},\ (3)\ J^2\left(\frac{\partial}{\partial x_3}\right)=-\frac{\partial}{\partial x_3},\ (4)\ J^2\left(\frac{\partial}{\partial y_1}\right)=-\frac{\partial}{\partial y_1},\ (5)\ J^2\left(\frac{\partial}{\partial y_2}\right)=-\frac{\partial}{\partial y_2},\ (6)\ J^2\left(\frac{\partial}{\partial y_3}\right)=-\frac{\partial}{\partial y_3}. \tag{5}$$

As clear from the above $J^2=-I$ are complex structures. In this study, above holomorpfic structures will be used.

5. Lagranian Dynamical Equation and System

Definition 5: [13,14,15]: Let M be an $n-$dimensional manifold also TM its tangent bundle with canonical projection $\tau_M : TM \to TM$. TM is called the phase space of velocities of the base manifold M . Let $L: TM \to \mathrm{R}$ be a differentiable function on TM called the Lagrangian function . Where, $L=T-V$, T is kinetic energy and V is potential energy. We consider the closed $2-$form on TM given by

$$\Phi_L = -dd_J L. \tag{6}$$

Consider the equation

$$i_x \Phi_L = dE_L. \tag{7}$$

Then X is a vector field and i_x is reduction function that it is $i_x\Phi_L=\Phi_L(X)$. We shall see that (7) under a certain condition on X is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as **Lagrange dynamical equation**. We shall see that for motion in a potential,

$$E_L = V(L) - L \tag{8}$$

is an energy function and $V = JX$ a Liouville vector field. Here dE_L denotes the differential of E_L. The triple (TM, Φ_L, X) is known as Euler-Lagrangian system on the tangent bundle

TM . If it is continued the operations on (7) for any coordinate system $\left(q^i(t), p_i(t)\right)$, infinite dimension Lagrange's equation is obtained the form below:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)-\frac{\partial L}{\partial q^i}=0, \quad \frac{\partial q^i}{\partial t}=\dot{q}^i, \quad i=1,...,n. \tag{9}$$

6. Holonomy, Riemannian and Flat Manifolds

The holonomy of a connection on a smooth manifold is a general geometrical consequence of the curvature of the connection measuring the extent to that parallel transport around closed loops fails to preserve the geometrical data being transported. For flat connections, the associated holonomy is a type of monodromy, and is an naturally global notion. For curved connections, holonomy has nontrivial local and global features. A (smooth) Riemannian manifold or (smooth) Riemannian space (M, g) is a real smooth manifold M equipped with an inner product g on the tangent space T_pM at each point p that varies smoothly from point to point in the sense that if X and Y are vector fields on M, then $p \mapsto g_p\left(X(p), Y(p)\right)$ is a smooth function. A Riemannian manifold is said to be flat if its curvature is everywhere zero. Intuitively, a flat manifold is one that locally looks like Euclidean space in terms of distances and angles, e.g. the interior angles of a triangle add up to 180°.

7. Lagrangian Equations on Flat Manifolds

Now, we get Euler-Lagrange equations for quantum and classical mechanics on Flat manifolds. Firstly, let X be the vector field decided by

$$X=X^1\frac{\partial}{\partial x_1}+X^2\frac{\partial}{\partial x_2}+X^3\frac{\partial}{\partial x_3}+Y^1\frac{\partial}{\partial y_1}+Y^2\frac{\partial}{\partial y_2}+Y^3\frac{\partial}{\partial y_3}. \tag{10}$$

The vector field described by

$$V=J(X)=J\left(X^1\frac{\partial}{\partial x_1}+X^2\frac{\partial}{\partial x_2}+X^3\frac{\partial}{\partial x_3}+Y^1\frac{\partial}{\partial y_1}+Y^2\frac{\partial}{\partial y_2}+Y^3\frac{\partial}{\partial y_3}\right) \tag{11}$$

is said to be Liouville vector field.

$$d=\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}x_2+\frac{\partial}{\partial x_3}x_3+\frac{\partial}{\partial y_1}dy_1+\frac{\partial}{\partial y_2}dy_2+\frac{\partial}{\partial y_3}dy_3,\ F(M)\to\wedge' M\ . \tag{12}$$

The closed 2-form which is given by $\Phi_L=-dd_JL$ so that

$$\begin{aligned} d_J=&\left(\cos(x_3)\frac{\partial}{\partial y_1}+\sin(x_3)\frac{\partial}{\partial y_2}\right)dx_1+\left(-\sin(x_3)\frac{\partial}{\partial y_1}+\cos(x_3)\frac{\partial}{\partial y_2}\right)dx_2+\frac{\partial}{\partial y_3}dx_3\\ &+\left(-\cos(x_3)\frac{\partial}{\partial x_1}+\sin(x_3)\frac{\partial}{\partial x_2}\right)dy_1+\left(-\sin(x_3)\frac{\partial}{\partial x_1}-\cos(x_3)\frac{\partial}{\partial x_2}\right)dy_2-\frac{\partial}{\partial x_3}dy_3. \end{aligned} \tag{13}$$

The above equation by adding L (Lagrange function) impact has $d_J L$. $\Phi_L = -dd_J L$ has the $d_J L$ with the differential.

$$1.\ f \wedge g = -g \wedge f,\ 2.\ f \wedge g(v) = f(v)g - g(v)f,\ 3.\ \frac{\partial x}{\partial x} = \delta_x^x = 1,\ \frac{\partial x}{\partial y} = \delta_y^x = 0, \qquad (14)$$

considering the external product and kronecker delta features that then we calculate

$$i_x\Phi_L = \Phi_L(X) =$$

$$+X^1\left[\begin{array}{l}
\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}[dx_1-dx_1]+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}[dx_1-dx_1]+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_3+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_3 \\
-\frac{\partial^2 L}{\partial x_1\partial y_3}dx_3-\sin(x_3)\frac{\partial L}{\partial y_1}dx_3+\cos(x_3)\frac{\partial L}{\partial y_2}dx_3+\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dy_1-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dy_1 \\
+\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dy_2 \\
+\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dy_2+\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_2 \\
+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}dx_2-\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}dx_2+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dy_3+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dy_3+\frac{\partial^2 L}{\partial x_1\partial x_3}dy_3
\end{array}\right]$$

$$+X^2\left[\begin{array}{l}
-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}dx_1+\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_2 \\
+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_2+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_2-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_2-\frac{\partial^2 L}{\partial x_2\partial y_3}dx_3-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_3 \\
+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_3-\cos(x_3)\frac{\partial L}{\partial y_1}dx_3+\cos(x_3)\frac{\partial L}{\partial y_2}dx_3+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dy_1-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dy_1 \\
-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dy_1+\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dy_2 \\
-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dy_2
\end{array}\right]$$

$$+X^3\left[\begin{array}{l}
-\frac{\partial^2 L}{\partial x_1\partial y_3}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_1-\frac{\partial^2 L}{\partial x_2\partial y_3}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_2 \\
-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_2+\frac{\partial^2 L}{\partial x_3\partial y_3}[dx_3-dx_3]-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dx_3+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dx_3+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dy_1 \\
-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dy_1+\frac{\partial^2 L}{\partial y_1\partial y_3}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dy_2+\frac{\partial^2 L}{\partial y_2\partial y_3}dy_2
\end{array}\right]$$

$$+Y^1\left[\begin{array}{l}
-\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dx_1+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dx_1 \\
-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dx_2-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dx_2-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dx_2 \\
-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dx_3+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dx_3-\frac{\partial^2 L}{\partial y_1\partial y_3}dx_3+\sin(x_3)\frac{\partial L}{\partial x_1}dx_3+\cos(x_3)\frac{\partial L}{\partial x_2}dx_3 \\
-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial x_1}[dy_1-dy_1]+\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial x_2}[dy_1-dy_1]+\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial x_2}dy_2 \\
-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial x_1}dy_2+\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial x_2}dy_2+\frac{\partial^2 L}{\partial y_1\partial x_3}dy_3-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_3+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_3
\end{array}\right]$$

$$
\begin{aligned}
&+Y^2\left[\begin{array}{l}
-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dx_1-\frac{\partial^2 L}{\partial x_2\partial y_3}dx_3\\
-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dx_3-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dx_3-\frac{\partial^2 L}{\partial y_2\partial y_3}dx_3-\cos(x_3)\frac{\partial L}{\partial x_1}dx_3+\sin(x_3)\frac{\partial L}{\partial x_2}dx_3-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dx_2\\
-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dx_2-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dx_2+\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial x_1}dy_1-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial x_2}dy_1\\
-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial x_1}dy_1-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial x_2}dy_1-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial x_1}\left[dy_2-dy_2\right]-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial x_2}\left[dy_2-dy_2\right]+\frac{\partial^2 L}{\partial y_2\partial x_3}dy_3\\
-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_3-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_3
\end{array}\right]\\
&+Y^3\left[\begin{array}{l}
-\frac{\partial^2 L}{\partial x_1\partial x_3}dx_1-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dx_1-\frac{\partial^2 L}{\partial x_2\partial x_3}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dx_2\\
-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dx_2-\frac{\partial^2 L}{\partial x_3\partial x_3}dx_3-\frac{\partial^2 L}{\partial y_3\partial y_3}dx_3-\frac{\partial^2 L}{\partial y_3\partial x_3}\left[dy_3-dy_3\right]-\frac{\partial^2 L}{\partial y_2\partial x_3}dy_1+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_1\\
-\frac{\partial^2 L}{\partial y_1\partial x_3}dy_1+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_1-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_2-\frac{\partial^2 L}{\partial y_3\partial x_3}dy_2
\end{array}\right]
\end{aligned}
\tag{15}
$$

Energy function and its differential are like the following:

$$dE_L = d\left[V(L)-L\right] \tag{16}$$

This differential form as follows:

$$
\begin{aligned}
dE_L = {}& X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}dx_1+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}dx_1\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}dx_1+\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}dx_1\right]+X^3\frac{\partial^2 L}{\partial x_1\partial y_3}dx_1\\
&+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dx_1+\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dx_1\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial x_1}dx_1+\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial x_2}dx_1\right]-Y^3\frac{\partial^2 L}{\partial x_1\partial x_3}dx_1-\frac{\partial L}{\partial x_1}dx_1\\
&+X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_2\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_2+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_2\right]+X^3\frac{\partial^2 L}{\partial x_2\partial y_3}dx_2\\
&+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dx_2+\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dx_2\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial x_1}dx_2+\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial x_2}dx_2\right]-Y^3\frac{\partial^2 L}{\partial x_2\partial x_3}dx_2-\frac{\partial L}{\partial x_2}dx_2\\
&+X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_3+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_3\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_3+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_3\right]+X^3\frac{\partial^2 L}{\partial x_3\partial y_3}dx_3\\
&+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dx_3+\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dx_3\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial x_1}dx_3+\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial x_2}dx_3\right]-Y^3\frac{\partial^2 L}{\partial x_3\partial x_3}dx_3-\frac{\partial L}{\partial x_3}dx_3\\
&+X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dy_1\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dy_1+\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dy_1\right]+X^3\frac{\partial^2 L}{\partial y_1\partial y_3}dy_1\\
&+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial x_1}dy_1+\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial x_2}dy_1\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial x_1}dy_1+\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial x_2}dy_1\right]-Y^3\frac{\partial^2 L}{\partial y_1\partial x_3}dy_1-\frac{\partial L}{\partial y_1}dy_1\\
&+X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dy_2+\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dy_2\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dy_2\right]+X^3\frac{\partial^2 L}{\partial y_2\partial y_3}dy_2\\
&+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial x_1}dy_2+\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial x_2}dy_2\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial x_1}dy_2+\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial x_2}dy_2\right]-Y^3\frac{\partial^2 L}{\partial y_2\partial x_3}dy_2-\frac{\partial L}{\partial y_2}dy_2
\end{aligned}
$$

$$+X^1\left[\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dy_3+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dy_3\right]+X^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dy_3+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dy_3\right]+X^3\frac{\partial^2 L}{\partial y_3\partial y_3}dy_3$$
$$+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_3+\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_3\right]+Y^2\left[-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial x_1}dy_3+\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial x_2}dy_3\right]-Y^3\frac{\partial^2 L}{\partial y_3\partial x_3}dy_3-\frac{\partial L}{\partial y_3}dy_3. \tag{17}$$

If we use (7) we obtain the equations given by

$$\begin{aligned}&X^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_1\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_1\partial y_2}dx_1\right]+X^2\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_2\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_2\partial y_2}dx_1\right]\\&+X^3\left[-\cos(x_3)\frac{\partial^2 L}{\partial x_3\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial x_3\partial y_2}dx_1\right]+Y^1\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_1\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_1\partial y_2}dx_1\right]\\&+Y^2\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_2\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_2\partial y_2}dx_1\right]+Y^3\left[-\cos(x_3)\frac{\partial^2 L}{\partial y_3\partial y_1}dx_1-\sin(x_3)\frac{\partial^2 L}{\partial y_3\partial y_2}dx_1\right]=-\frac{\partial L}{\partial x_1}dx_1\end{aligned} \tag{18}$$

simplified version

$$\begin{aligned}&-\cos(x_3)\left[X^1\frac{\partial}{\partial x_1}+X^2\frac{\partial}{\partial x_2}+X^3\frac{\partial}{\partial x_3}+Y^1\frac{\partial}{\partial y_1}+Y^2\frac{\partial}{\partial y_2}+Y^3\frac{\partial}{\partial y_3}\right]\frac{\partial L}{\partial y_1}\\&-\sin(x_3)\left[X^1\frac{\partial}{\partial x_1}+X^2\frac{\partial}{\partial x_2}+X^3\frac{\partial}{\partial x_3}+Y^1\frac{\partial}{\partial y_1}+Y^2\frac{\partial}{\partial y_2}+Y^3\frac{\partial}{\partial y_3}\right]\frac{\partial L}{\partial y_2}=-\frac{\partial L}{\partial x_1}.\end{aligned} \tag{19}$$

or

$$-\cos(x_3)X\left(\frac{\partial L}{\partial y_1}\right)-\sin(x_3)X\left(\frac{\partial L}{\partial y_2}\right)=-\frac{\partial L}{\partial x_1}. \tag{20}$$

Considering the curve α , an integral curve of X i.e. $X(\alpha(t))=\frac{\partial\alpha}{\partial t}$, we can find the equations as follows: $-\cos(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_1}\right)-\sin(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_2}\right)+\frac{\partial L}{\partial x_1}=0$. In other equation, we are doing similar operations. In the result of this process, we get the following equations.

$$\begin{aligned}&(1)\ -\cos(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_1}\right)-\sin(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_2}\right)+\frac{\partial L}{\partial x_1}=0, && (2)\ \sin(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_1}\right)-\cos(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_2}\right)+\frac{\partial L}{\partial x_2}=0,\\&(3)-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_3}\right)+\frac{\partial L}{\partial x_3}=0, && (4)\cos(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_1}\right)-\sin(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_2}\right)+\frac{\partial L}{\partial y_1}=0,\\&(5)\ \sin(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_1}\right)+\cos(x_3)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_2}\right)+\frac{\partial L}{\partial y_2}=0, && (6)\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_3}\right)+\frac{\partial L}{\partial y_3}=0.\end{aligned} \tag{21}$$

so that these equations (21) are called ***Euler–Lagrange equations constructed on Flat manifolds***. Thus the triple (TM,Φ_L,X) is named as a *Euler–Lagrange mechanical system* on Flat manifolds.

8. Equations Closed Solution

These partial differential equations (21) are depending on time. We can solve these equations using the symbolic computational program. The software codes and solutions of these equations as follows:

(1) $>PDEL_1:=-cos(x_3)*diff(diff(L_1(x_1,x_2,x_3,y_1,y_2,y_3,t),y_1),t)-$
$sin(x_3)*diff(diff(L_1(x_1,x_2,x_3,y_1,y_2,y_3,t),y_2),t)+diff(L_1(x_1,x_2,x_3,y_1,y_2,y_3,t),x_1);$
$>answer:=pdsolve(PDEL_1);$
$>answer:=(L_1(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_1(x_1)*F_4(y_1)*F_5(y_2)*F_7(t)*F_9(x_3)*F_8(x_2,y_3),$
where; $\{-c_3*F_8(x_2,y_3)=0,F_9(x_3)=c_3/(cos(x_3)*c_4*c_7+sin(x_3)*c_5*c_7-c_1),$
$diff(F_1(x_1),x_1)=c_1*F_1(x_1),diff(F_4(y_1),y_1)=c_4*F_4(y_1),diff(F_5(y_2),y_2)=c_5*F_5(y_2),$
$diff(F_7(t),t)=c_7*F_7(t)\}.$

(2) $>PDEL_2:=sin(x_3)*diff(diff(L_2(x_1,x_2,x_3,y_1,y_2,y_3,t),y_1),t)-$
$cos(x_3)*diff(diff(L_2(x_1,x_2,x_3,y_1,y_2,y_3,t),y_2),t)+diff(L_2(x_1,x_2,x_3,y_1,y_2,y_3,t),x_2);$
$>answer:=pdsolve(PDEL_2);$
$>answer:=(L_2(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_2(x_2)*F_4(y_1)*F_5(y_2)*F_7(t)*F_9(x_3)*F_8(x_1,y_3),$
where; $\{-c_3*F_8(x_1,y_3)=0,F_9(x_3)=c_3/(-sin(x_3)*c_4*c_7+cos(x_3)*c_5*c_7-c_2),$
$diff(F_2(x_2),x_2)=c_2*F_2(x_2),diff(F_4(y_1),y_1)=c_4*F_4(y_1),diff(F_5(y_2),y_2)=c_5*F_5(y_2),$
$diff(F_7(t),t)=c_7*F_7(t)\}.$

(3) $>PDEL_3:=-diff(diff(L_3(x_1,x_2,x_3,y_1,y_2,y_3,t),y_3),t)+diff(L_3(x_1,x_2,x_3,y_1,y_2,y_3,t),x_3);$
$>answer:=pdsolve(PDEL_3);$
$>answer:=(L_3(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_3(x_3)*F_6(y_3)*F_7(t)*F_8(x_1,x_2,y_1,y_2),$
where; $\{F_8(x_1,x_2,y_1,y_2)*c_7=0,\ diff(F_3(x_3),x_3)=c_3*F_3(x_3),\ diff(F_6(y_3),y_3)=c_6*F_6(y_3),$
$diff(F_7(t),t)=c_3*F_7(t)/c_6-c_7/c_6\}.$

(4) $>PDEL_4:=cos(x_3)*diff(diff(L_4(x_1,x_2,x_3,y_1,y_2,y_3,t),x_1),t)-$
$sin(x_3)*diff(diff(L_4(x_1,x_2,x_3,y_1,y_2,y_3,t),x_2),t)+diff(L_4(x_1,x_2,x_3,y_1,y_2,y_3,t),y_1);$
$>answer:=pdsolve(PDEL_4);$
$>answer:=(L_4(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_1(x_1)*F_2(x_2)*F_4(y_1)*F_7(t)*F_8(y_2,y_3),$
where; $\{-c_3*F_8(y_2,y_3)=0,\ F_9(x_3)=c_3/(cos(x_3)*c_1*c_7-sin(x_3)*c_2*c_7+c_4),$
$diff(F_1(x_1),x_1)=c_1*F_1(x_1),\ diff(F_2(x_2),x_2)=c_2*F_2(x_2),\ diff(F_4(y_1),y_1)=c_4*F_4(y_1),$
$diff(F_7(t),t)=c_7*F_7(t)\}.$

(5) $>PDEL_5:=sin(x_3)*diff(diff(L_5(x_1,x_2,x_3,y_1,y_2,y_3,t),x_1),t)+$
$cos(x_3)*diff(diff(L_5(x_1,x_2,x_3,y_1,y_2,y_3,t),x_2),t)+diff(L_5(x_1,x_2,x_3,y_1,y_2,y_3,t),y_2);$
$>answer:=pdsolve(PDEL_5);$
$>answer:=(L_5(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_1(x_1)*F_2(x_2)*F_5(y_2)*F_7(t)*F_9(x_3)*F_8(y_1,y_3),$
where; $\{-c_3*F_8(y_1,y_3)=0,\ F_9(x_3)=c_3/(sin(x_3)*c_1*c_7+cos(x_3)*c_2*c_7+c_2),$
$diff(F_1(x_1),x_1)=c_1*F_1(x_1),\ diff(F_2(x_2),x_2)=c_2*F_2(x_2),\ diff(F_5(y_2),y_2)=c_5*F_5(y_2),$
$diff(F_7(t),t)=c_7*F_7(t)\}.$

(6) $>PDEL_6:=diff(diff(L_6(x_1,x_2,x_3,y_1,y_2,y_3,t),x_3),t)+diff(L_6(x_1,x_2,x_3,y_1,y_2,y_3,t),y_3);$
$>answer:=pdsolve(PDEL_6);$
$>answer:=(L_6(x_1,x_2,x_3,y_1,y_2,y_3,t)=F_3(x_3)*F_6(y_3)*F_7(t)*F_8(x_1,x_2,y_1,y_2),$
where; $\{F_8(x_1,x_2,y_1,y_2)*c_7=0,\ diff(F_3(x_3),x_3)=c_3*F_3(x_3),\ diff(F_6(y_3),y_3)=c_6*F_6(y_3),$
$diff(F_7(t),t)=(c_7-F_7(t)*c_6)/c_3\}.$

(22)

9. Discussion

It is well known that classical field theory utilizes traditionally the language of Lagrangian dynamics such that this theory was extended to time-dependent classical mechanics. A Lagrange space has been certified as an excellent model for some important problems in relativity, gauge theory, and electromagnetism such that it gives a model for both the gravitational and electromagnetic field in a very natural blending of the geometrical structures of the space with the characteristic properties of these physical fields.

Euler-Lagrangian dynamics is used as a model for field theory, quantum physics, optimal control, biology and fluid dynamics. Most important advantage of flat manifold is to allow the calculation of linear distance. The obtained equations (21) on Flat manifolds are very important to explain the rotational spatial mechanical-physical problems. In addition in the equations, using the symbolic computational program, closed solutions (22) were found. They has been used in solving problems in different physical and mechanical area and easily seen extremely useful in applications.

For future research, Euler–Lagrange equations constructed on Flat manifolds are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics [16,17,18].

References

[1] Kasap, Z., Weyl-Mechanical Systems on Tangent Manifolds of Constant W-Sectional Curvature, Int. J. Geom. Methods Mod. Phys. Vol.10, No.10, 1-13, 2013.

[2] Kapovich, M., Flat Conformal Structures on 3-Manifolds, I: Uniformization of Closed Seifert Manifolds, J. Differential Geometry, 38, 191-215, 1993.

[3] Schwartz, F., Volumetric Penrose Inequality for Conformally Flat Manifolds, Annales Henri Poincaré, Volume 12, Issue 1, 67-76, 2011.

[4] Kulkarni, R.S., Conformally Flat Manifolds, Proc. Nat. Acad. Sci. Vol.69, No.9, 2675-2676, 1972.

[5] Dotti, I.G. and Miatello, R.J., On The Cohomology Ring of Flat Manifolds with A Special Structure, Revista De La Uni On Matematica Argentina, Volumen 46, Numero 2, 133–147, 2005.

[6] Szczepanski, A., Kähler at Manifolds of Low Dimensions, Institut des Hautes Etudes Scientifiques, 2005.

[7] Ge, Y., Wang, G. and Wu, J., The Mass of An Asymptotically Flat Manifold, arXiv:1211.3645v2 , 2012.

[8] Gonzalez, M.D.M., Singular Sets of A Class of Locally Conformally Flat Manifolds, Duke Mathematical Journal, Vol.129, No.3, 2005.

[9] Akbulut, S. and Kalafat, M., A Class of Locally Conformally Flat 4-Manifolds, arXiv:0807.0837v5.

[10] http://en.wikipedia.org/wiki/Pseudoholomorphic_curve.

[11] Newlander, A. and Nirenberg, L., Complex Analytic Coordinates in Almost Complex Manifolds. Ann. of Math, 391-404, 65, 1957.

[12] Brozos-Vazquez, M., Gilkey, P. and Merino, E., Geometric Realizations of Kaehler and of Para-Kaehler Curvature Models, IJGMMP, Vol.7, No.3, 505–515, 2010.

[13] Klein, J., Escapes Variationnals et Mécanique, Ann. Inst. Fourier, Grenoble, 12, 1-124, 1962.

[14] De Leon, M. and Rodrigues, P.R., Methods of Differential Geometry in Analytical Mechanics, North-Hol. Math. St., 152, 263-397, 1989.

[15] Abraham, R., Marsden, J.E., Ratiu, T., Manifolds Tensor Analysis and Applications, Springer, 483-542, 2001.

[16] Weyl, H., Space-Time-Matter, Dover Publ. 1922.Translated from the 4th German Edition by H. Brose.London: Methuen. Reprint New York: Dover, 1952.

[17] Miron, R., Hrimiuc, D., Shimada, H. and Sabau, S.V., The Geometry of Hamilton and Lagrange Spaces, eBook ISBN: 0-306-47135-3, Kluwer Academic Publishers, New York, 2002.

[18] De León, M. and Rodrigues, P.R., Generalized Classical Mechanics and Field Theory, North-Holland Mathematics Studies 112, North-Holland, Amsterdam, 1985.