BALKANJM 01 (2013) 61-71

Contents lists available at BALKANJM

BALKAN JOURNAL OF MATHEMATICS

journal homepage: www.balkanjm.com

Review of Time Domain Waveguide Modes in Perspective of Evolutionary Approach to Electromagnetics (EAE)

Ummu Sahin Sener[*a] and Ersin Sener[b]

[a,b] *Department of Mathematics, Kirklareli University, Kirklareli, Turkey*

ARTICLE INFO

Article history:
Received 30 December 2012
Accepted 1 February 2013
Available online 6 February 2013

Keywords:
Maxwell equations
Time domain waveguide mode
Klein-Gordon equation
Neumann-Dirichlet problems

ABSTRACT

A problem for electromagnetic fields is considered by a direct analytical time-domain method called *Evolutionary Approach to Electromagnetics* (EAE). The problem is solved analytically in compliance with a causality principle. The method for time-domain modes has brought a breath of fresh. EAE method is most useful and available one by means of ∂_t time derivative. The method was developed at the end of 80s by O.A. Tretyakov.
This study is distinguished a few studies via the method in the last decades. The studies are demonstrated for time-domain waveguide modes.

1. Introduction

Time-harmonic waveguide modes are usually interpreted for signal transmission along waveguides. However, this model has two essential physical drawbacks. Firstly, the time-harmonic signals are non-casual. It means that their propagation starts at time $t=-\infty$ and continues up to time $t=\infty$. Secondly, these signals have frequency bandwidth equal to zero. Therefore this is not a satisfactory model for the signal transmission problems in the waveguides. Leaving aside the works based on the synthesizing realistic signals via continual superposition of the time-harmonic waves with using Fourier transform or Laplace transform, it seems that one of the first noticeable approaches for direct time domain solutions of electromagnetic problems was developed within the framework of four-dimensional relativistic formalism in electrodynamics [1].

[*] Corresponding author: *E-mail*: ummusahin@gmail.com (U., S., Sener).

Then another alternative approach called the *Evolutionary Approach to Electromagnetics* (EAE) suggested in 80s was proposed for the direct time-domain theory of the cavity and waveguide modes. Important published works regarded to the EAE method for the time-domain waveguide problems were given in the literature [2]-[8]. The other set of important publications on this topic is based on the different techniques [9]-[13]. Knowledge about the properties of hyperbolic kind Partial Differential Equation (PDE) suggests alternative attacks to new classes of the waveguide problems. The proposed approach leads to PDEs for the modal amplitudes in the time-domain [1].

EAE method proposes analytical solution. Maxwell's equation system is solved analytically in time-domain via evolution equations that have been kept ∂_t and Laplacian. Electromagnetic fields are resolved from Maxwell's equation system in time-domain. The solution is obtained by investigating along waveguide.

2. Formulation of problem

A hollow (i.e., medium-free) waveguide with its cross-section domain S bounded by a closed singly connected contour L is considered. It is supposed that L has enough smooth shape which implies that none of possible *inner* angles of L (i.e., being measured within S) exceeds π and the cross section S maintains its form and size along the waveguide axis Oz [1].

Our aim is to solve the modal fields for the *TE* and *TM* modes which are a particular solution to the system of Maxwell's equations with the time derivative given as

$$\nabla\times\mathbf{E}(\mathbf{R},t) = -\mu_0\frac{\partial}{\partial t}\mathbf{H}(\mathbf{R},t) \quad , \quad \nabla\times\mathbf{H}(\mathbf{R},t) = \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}(\mathbf{R},t) \tag{1}$$

where $\mathbf{E}(\mathbf{R},t)$ and $\mathbf{H}(\mathbf{R},t)$ are the electric and magnetic fields, respectively. ε_0 and μ_0 are dielectric and magnetic constants for free-space, respectively. Because the fields will be excited by an initial condition technique, the source term is not considered in the Maxwell's equations. The vector $\mathbf{R}$ within the waveguide volume denotes an observation point. t is observation time. Let's introduce a right-handed triplet of the mutually orthogonal unit vectors ($\mathbf{z}$, $\boldsymbol{l}$, $\boldsymbol{n}$) where $\mathbf{z}\times\boldsymbol{l} = \boldsymbol{n}$. The unit vector $\mathbf{z}$ and $\boldsymbol{l}$ are tangential to the axis Oz and contour L, respectively. The unit vector $\boldsymbol{n}$ is outward normal to the cross-section of domain S.

Let's decompose the vector $\mathbf{R}$ and Nabla operator ∇ onto their transverse and longitudinal parts as

$$\mathbf{R}=\mathbf{r}+\mathbf{z}z, \qquad \nabla=\nabla_{\perp}+\mathbf{z}\partial_z \tag{2}$$

where the projection $\mathbf{r}$ is a position vector within the domain S and $\nabla_{\perp}$ is the transverse Laplacian operator [1].

Subject of our study is *real-valued* electromagnetic fields specified by the electric and magnetic field strength vectors $\mathbf{E}_m(\mathbf{R},t)$ and $\mathbf{H}_m(\mathbf{R},t)$, respectively. Separate these vectors onto their transverse and longitudinal parts similarly to performed in Eq. (2), i.e.,

$$\begin{aligned}\mathbf{E}_m(\mathbf{R},t)&=\mathbf{E}(\mathbf{r},z,t)+\mathbf{z}E_z(\mathbf{r},z,t)\\ \mathbf{H}_m(\mathbf{R},t)&=\mathbf{H}(\mathbf{r},z,t)+\mathbf{z}H_z(\mathbf{r},z,t)\end{aligned} \tag{3}$$

where $m=1,2,\ldots$. Because the waveguide surface is supposed to have physical properties of the perfect electric conductor, the following boundary conditions hold over the waveguide surface [1]

$$\mathbf{n}\cdot\mathbf{H}_m(\mathbf{R},t)\big|_L=0 \quad , \quad \mathbf{l}\cdot\mathbf{E}_m(\mathbf{R},t)\big|_L=0 \quad , \quad \mathbf{z}\cdot\mathbf{E}_m(\mathbf{R},t)\big|_L=0 \tag{4}$$

2.1. Neumann and Dirichlet boundary eigenvalue problem

Let's consider the Neumann boundary eigenvalue problem for *TE* time-domain modes for the operator $\nabla_{\perp}^2$ as

$$\left(\nabla_{\perp}^2+\upsilon_m^2\right)\psi_m(\mathbf{r})=0 \quad , \quad \left.\frac{\partial\psi_m(\mathbf{r})}{\partial n}\right|_L=0 \quad , \quad \frac{\upsilon_m^2}{S}\int_S\left|\psi_m(\mathbf{r})\right|^2 ds=1\,N \tag{5}$$

where $\partial_n=\mathbf{n}\cdot\nabla_{\perp}$ is the normal derivative on the contour L. $\upsilon_m^2>0$ and $m=1,2,3,..$ are the eigenvalues and their regulation numbers of position on a real axis in the increasing order of their numerical values. The potentials $\psi_m(\mathbf{r})$ are the eigenvectors of the corresponding eigenvalues. Force dimension N (i.e., newton) in Eq. (5) is involved in order to provide the required physical dimensions for the field vector components of $\mathbf{E}_m$ and $\mathbf{H}_m$ as Vm^{-1} and Am^{-1}, respectively [1], [8].

For eigenvalue $\upsilon_0^2=0$, the problem (5) will have the following form

$$\nabla_{\perp}^2\psi_0(\mathbf{r})=0 \quad , \quad \left.\frac{\partial\psi_0(\mathbf{r})}{\partial n}\right|_L=0 \tag{6}$$

where the function $\psi_0(\mathbf{r})$ is a harmonic function and its value is distinct from *zero*. The minimum-maximum theorem for the harmonic functions yields that $\psi_0(\mathbf{r}) = C$ where $r \in L + S$ and C is an arbitrary constant [1].

Every particular solution $\psi_m(\mathbf{r})$ to the Neumann problem (5) generates the *TE* time-domain modal fields with the components as

$$\begin{aligned}
&\mathbf{E}^h_{zm} = 0\\
&\upsilon_m^{-1}\mathbf{E}^h_m = \left\langle -\partial_{\upsilon_m ct} h_m(z,t)\right\rangle \left[\varepsilon_0^{-\frac{1}{2}} A_m^{TE} \nabla_\perp \psi_m(\mathbf{r}) \times \mathbf{z}\right]\\
&\upsilon_m^{-1}\mathbf{H}^h_m = \left\langle \partial_{\upsilon_m z} h_m(z,t)\right\rangle \left[\mu_0^{-\frac{1}{2}} A_m^{TE} \nabla_\perp \psi_m(\mathbf{r})\right]\\
&\upsilon_m^{-1}\mathbf{H}^h_{zm} = \left\langle h_m(z,t)\right\rangle \left[\mu_0^{-\frac{1}{2}} \upsilon_m A_m^{TE} \psi_m(\mathbf{r})\right]
\end{aligned} \tag{7}$$

where $\partial_{\upsilon_m ct} = (1/c\upsilon_m)\partial/\partial t$, $\partial_{\upsilon_m z} = (1/\upsilon_m)\partial/\partial z$ and $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Specially, the potential $\psi_0(\mathbf{r})$ generates a one-component modal field as

$$\mathbf{E}_0(\mathbf{r},z,t) = 0 \quad , \quad \mathbf{H}_0(\mathbf{r},z,t) = \mathbf{z}C \tag{8}$$

where dimension Am^{-1} should be assigned to constant C [1], [8].

The potential $h_m(z,t)$ in Eq. (7) is governed by Klein-Gordon Equation (KGE)

$$\left(\partial^2_{\upsilon_m ct} - \partial^2_{\upsilon_m z} + 1\right) h_m(z,t) = 0 \tag{9}$$

which is known as a *generalized* wave equation [1], [5], [8].

As similar to the problem of the *TE* time-domain modes, the Dirichlet boundary eigenvalue problem for *TM* time-domain modes for the operator $\nabla^2_\perp$ can be stated as follows

$$\left(\nabla^2_\perp + \kappa^2_m\right)\phi_m(\mathbf{r}) = 0 \quad , \quad \phi_m(\mathbf{r})\big|_L = 0 \quad , \quad \frac{\kappa^2_m}{S}\int_S \left|\phi_m(\mathbf{r})\right|^2 ds = 1\,N \tag{10}$$

where $\kappa^2_m > 0$, $m = 1,2,3,..$ are the eigenvalues. The potential $\phi_0(\mathbf{r})$ will be zero [1].

The solution $\phi_m(\mathbf{r})$ to the Dirichlet problem (10) generates the *TM* time-domain modal fields with the following components

$$
\begin{aligned}
&\mathbf{H}^{e}_{zm} = 0\\
&\kappa_m^{-1}\mathbf{H}^{e}_{m} = \left\langle -\partial_{\kappa_m ct} e_m(z,t)\right\rangle \left[\mathbf{z}\times \mu_0^{-\frac{1}{2}}\kappa_m A_m^{TM}\nabla_\perp \phi_m(\mathbf{r})\right]\\
&\kappa_m^{-1}\mathbf{E}^{e}_{m} = \left\langle \partial_{\kappa_m z} e_m(z,t)\right\rangle \left[\varepsilon_0^{-\frac{1}{2}}\kappa_m A_m^{TM}\nabla_\perp \phi_m(\mathbf{r})\right]\\
&\kappa_m^{-1}\mathbf{E}^{e}_{zm} = \left\langle e_m(z,t)\right\rangle \left[\varepsilon_0^{-\frac{1}{2}}\kappa_m A_m^{TM} \phi_m(\mathbf{r})\right]
\end{aligned}
\tag{11}
$$

where $\partial_{\kappa_m ct} = (1/c\kappa_m)\partial/\partial t$, $\partial_{\kappa_m z} = (1/\kappa_m)\partial/\partial z$. The potential $e_m(z,t)$ generates the modal amplitudes in Eq. (11) is the solution of the KGE as

$$
\left(\partial^2_{\kappa_m ct} - \partial^2_{\kappa_m z} + 1\right)e_m(z,t) = 0 \tag{12}
$$

which is similar to Eq. (9) [1], [5].

The factors selected by the *square* brackets [.] in Eq. (7) and (11) describe the modal field patterns in the waveguide cross section. Their physical dimensions are Vm^{-1} and Am^{-1} for the electric and magnetic field components, respectively. The factors selected by the *broken* brackets $\langle . \rangle$ in Eq. (7) and (11) are dimensionless. Their physical sense is about the time-dependent modal amplitudes of appropriate modal field components [1], [8].

The set of the *TE* and *TM* modes (as the vector functions of transverse coordinates) is complete due to the completeness of their generating potentials in the same energetic space. The completeness comes from Sturm-Liouville and Weyl theorem in functional analysis about the orthogonal detachments of Hilbert space $L_2(S)$ [3]-[5]. This energetic space can be specified by an inner product as

$$
\left(X_1, X_2\right) = \frac{1}{S}\int_S \left(\varepsilon_0 \mathbf{E}_1.\mathbf{E}_2 + \mu_0 \mathbf{H}_1.\mathbf{H}_2\right) ds < \infty \tag{13}
$$

where $X_i = col(\mathbf{E}_i, \mathbf{H}_i)$, $i = 1,2,..$, col. stands for ”column”. One can verify that $\left(X_m^{TE}, X_n^{TM}\right) = 0$ for any combinations of *m* and *n* with the values $0,1,2,...$, independently. Therefore, any pair of the *TE* and *TM* time-domain modes is orthogonal in the sense of inner product (13) [1].

3. Modal amplitude problem

The KGE in Eq. (9) for the *TE* modes and the KGE in Eq. (12) for the *TM* modes have the same structure. After introducing the *scaled time* τ and *scaled coordinate* ξ as

$$\begin{aligned} &\tau = \upsilon_m ct \quad , \quad \xi = \upsilon_m z \quad for \;\; TE-\text{modes} \\ &\tau = \kappa_m ct \quad , \quad \xi = \kappa_m z \quad for \;\; TM-\text{modes}, \end{aligned} \tag{14}$$

the KGE in Eq. (9) and Eq. (12) can be written in the general form of

$$\left(\partial_\tau^2 - \partial_\xi^2 + 1\right) f(\xi,\tau) = 0 \tag{15}$$

where $f(\xi,\tau)$ is either $h_m(\xi,\tau)$ provided that $\xi = \kappa_m z$ and $\tau = \kappa_m ct$ or $e_m(\xi,\tau)$ provided that $\xi = \upsilon_m z$ and $\tau = \upsilon_m ct$ [1].

The KGE maintains its form under an action of a *Poincare group* within the framework of the *group theory*. In this aspect, Miller established eleven so called orbits of symmetry in terms of the *group theory* [14]. His results are crucial for development of the electromagnetic field theory in the time-domain [1].

On the basing of Miller's idea, let us interpret solution to the KGE in Eq. (15) as a function with a new arguments, namely: $f \equiv f(\xi,\tau) = f\left[u(\xi,\tau), v(\xi,\tau)\right]$. The "new" variables (u,v) are *unknown* yet, but suppose that they are twice differentiable functions of the "old" variables (ξ,τ). Substitution of $f\left[u(\xi,\tau), v(\xi,\tau)\right]$ as a formal solution to Eq. (15) yields a new form of this equation as

$$\begin{aligned} &\left[\left(\frac{\partial u}{\partial \tau}\right)^2 - \left(\frac{\partial u}{\partial \xi}\right)^2\right]\frac{\partial^2 f}{\partial u^2} + \left[\left(\frac{\partial v}{\partial \tau}\right)^2 - \left(\frac{\partial v}{\partial \xi}\right)^2\right]\frac{\partial^2 f}{\partial v^2} + \left[\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2}\right]\frac{\partial f}{\partial u} \\ &\qquad + \left[\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial \xi^2}\right]\frac{\partial f}{\partial v} + 2\left[\frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} - \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \xi}\right]\frac{\partial^2 f}{\partial u \partial v} + f = 0 \end{aligned} \tag{16}$$

where notice that the derivatives ∂_u and ∂_v act on the function $f(u,v)$ under study. The various combinations of the derivatives by ξ and τ of the functions $u(\xi,\tau)$ and $v(\xi,\tau)$ are appeared (unknown yet!) at the coefficients placed in square brackets. In order to solve Eq. (16), it is necessary to perform the following operations: a) Define the proper functions of u and v. b) Express the coefficients (placed in square brackets) as the functions of u and v.

After this step, Eq. (16) becomes a PDE with variable coefficients depending on *u* and *v*. c) Solve Eq. (16) via separation of the variables *u* and *v*. This can be done if and only if the functions of *u* and *v* are specified properly. For this aim, Miller obtained eleven pairs of inverse functions, i.e., $\xi(u,v)$ and $\tau(u,v)$ [14]. As an example, the first pair is $\xi = v$ and $\tau = u$ where $-\infty < u < \infty$, $-\infty < v < \infty$. In this case, $f(u,v)$ will be a product of the exponential functions and yields the time-harmonic waves [1].

In [1], is discussed case 2 ($\tau = u\cosh v$ and $\xi = u\sinh v$ with $-\infty \le u < \infty$, $-\infty < v < \infty$) from Millers' list. The discussion is underlined electromagnetic fields, energy and surplus of energy. The study says

$$f_\alpha(\xi,\tau) = \left(\frac{\tau-\xi}{\tau+\xi}\right)^{\frac{\alpha}{2}} J_\alpha\left(\sqrt{\tau^2-\xi^2}\right) \tag{17}$$

where the free parameter $\alpha > 0$. If α is integer, $J_\alpha\left(\sqrt{\tau^2-\xi^2}\right)$ is *cylindrical Bessel* function. If α is semi-integer $J_\alpha\left(\sqrt{\tau^2-\xi^2}\right)$ is *spherical Bessel* function.

Physically, Eq. (22) is the time dependent modal amplitude of the *longitudinal* field component and corresponds to $h_m(z,t)$ in Eq. (7) or $e_m(z,t)$ in Eq. (11) [1].

The analysis of the modal amplitudes of the *TE* and *TM* fields can be executed in *parallel* according to Eq. (7) and Eq. (11). The amplitudes of the *transverse* field components are presentable by the same formulas, namely [5], [8]

$$\begin{aligned} A(\xi,\tau) &= -\frac{\partial}{\partial\tau} f_\alpha(\xi,\tau) \equiv \left\langle -\partial_{\upsilon_m ct} h_m \right\rangle \equiv \left\langle -\partial_{\kappa_m ct} e_m \right\rangle \\ B(\xi,\tau) &= \frac{\partial}{\partial\xi} f_\alpha(\xi,\tau) \equiv \left\langle \partial_{\upsilon_m z} h_m \right\rangle \equiv \left\langle \partial_{\kappa_m z} e_m \right\rangle \end{aligned} \tag{18}$$

The modal amplitudes of the longitudinal components in Eq. (7) and (11) both are the solutions to the KGE (15). Then, the modal amplitudes of the *transverse* field components can be specified by Eq. (18). The direct differentiations of $f_\alpha(\xi,\tau)$ in accordance with these formulas result in

$$A_0(\xi,\tau)=\tau\frac{J_1\left(\sqrt{\tau^2-\xi^2}\right)}{\sqrt{\tau^2-\xi^2}} \quad , \quad B_0(\xi,\tau)=\xi\frac{J_1\left(\sqrt{\tau^2-\xi^2}\right)}{\sqrt{\tau^2-\xi^2}}$$

$$A_\alpha(\xi,\tau)=-\frac{\partial}{\partial\tau}f_\alpha=-\left(\frac{f_{\alpha-1}-f_{\alpha+1}}{2}\right) \quad , \quad B_\alpha(\xi,\tau)=\frac{\partial}{\partial\xi}f_\alpha=-\left(\frac{f_{\alpha-1}+f_{\alpha+1}}{2}\right) \tag{19}$$

where the free parameter $\alpha>0$.

In [8], is discussed case 5 ($\tau+\xi=2(u+v)$ and $\tau-\xi=(u-v)^2$ with $-\infty<u\,,\ v<\infty$) from Millers' list. The study says, modal amplitudes are found out explicitly and expressed via products of Airy functions with arguments dependent on *t* and *z*.

$$u+v=\frac{\tau+\xi}{2} \text{ and } u-v=\pm\sqrt{\tau-\xi}\,. \tag{20}$$

When we read the double sign $(\pm)$ as minus $(-)$, then

$$u=\frac{\tau+\xi}{4}-\frac{\sqrt{\tau-\xi}}{2} \text{ and } v=\frac{\tau+\xi}{4}+\frac{\sqrt{\tau-\xi}}{2}\,. \tag{21}$$

Calculations of the coefficients standing in the square brackets in (16) result in

$$[(\partial_\tau u)^2-(\partial_\xi u)^2]=-[(\partial_\tau v)^2-(\partial_\xi v)^2]=\frac{1}{4(u-v)}\,. \tag{22}$$

with simple calculation

$$\frac{\partial^2 f(u,v)}{\partial u^2}+4uf(u,v)=\frac{\partial^2 f(u,v)}{\partial v^2}+4vf(u,v)\,. \tag{23}$$

And with separation of variables $f(u,v)=U(u)V(v)$,

$$\frac{1}{U(u)}\frac{d^2U(u)}{du^2}+4u=\frac{1}{V(v)}\frac{d^2}{dv^2}V(v)+4v=4\alpha \tag{24}$$

At this point, it is convenient to slightly change notation for the variables *u* and *v*. [8] introduces the new *u* and *v* as

$$\bar{u}=\sqrt[3]{4}(\alpha-u) \text{ and } \bar{v}=\sqrt[3]{4}(\alpha-v)\,. \tag{25}$$

where α is a constant.

The final solution to KGE could be presented

$$f(\xi,\tau) = U(\bar{u})V(\bar{v}),\ 0 \le \xi \le \tau. \tag{26}$$

3.1. Initial condition for Klein-Gordon equation

The KGE (15) has to be supplemented with a pair of initial conditions. Physically, they specify the source signal for excitation. Suppose that such source is turned on at $t = 0$, however it does not act before. Then the initial conditions at $\xi = 0 \Rightarrow z = 0$ can be written as

$$f(\xi,\tau)\big|_{\xi=0} = \begin{Bmatrix} \phi(\tau) & , & \tau \ge 0 \Rightarrow t \ge 0 \\ 0 & , & \tau < 0 \Rightarrow t < 0 \end{Bmatrix} \quad , \quad \frac{\partial}{\partial \tau} f(\xi,\tau)\Big|_{\xi=0} = \begin{Bmatrix} \dot{\phi}(\tau) & , & \tau \ge 0 \Rightarrow t \ge 0 \\ 0 & , & \tau < 0 \Rightarrow t < 0 \end{Bmatrix}. \tag{27}$$

3. 2. The causality principle

The solution of the KGE have to be subjected for the physical requirements of the causality principle which can be interpreted in two ways: First, a *weak causality* condition states that all fields are *zero* before their sources are not turned on. In our case, this corresponds to $\tau < 0$ which relates to the initial condition. Second, a *strong causality* condition from the Einstein postulates that the electromagnetic field can not transfer energy more than the speed of the light c in the vacuum. In our case, this implies that the solution of the KGE should be *zero* beyond the distance $\xi = \tau$ (i.e., $z = ct$) which corresponds to the wave front of the electromagnetic wave. Thus, the solution of the KGE can be read physically as:

$$f(\xi,\tau) = \begin{Bmatrix} f(\xi,\tau) = 0 & , & \tau < 0 \\ f(\xi,\tau) \ne 0 & , & 0 \le \xi \le \tau \\ f(\xi,\tau) = 0 & , & \xi > \tau \end{Bmatrix}. \tag{28}$$

Eq. (17) and (26) are must obey eq. (27) and (28).

4. Conclusion

In this study, the time-domain waveguide modes are reminded analytically by a method of *Evolutionary Approach to Electromagnetics* (EAE). Especially, [1] for case 2 and [8] for case 5 are reconsidered. A time-dependent source function is thought overed in a waveguide with perfect electric conductor surface. In the future, the other possible solutions proposed from the Miller’s eleven cases will be considered for the solution of different waveguide mode problems.

Appendix

Complete list of substitutions $\xi(u,v)$ and $\tau(u,v)$ factorizing solution to Eq. (16) as $f(u,v)=U(u)V(v)$.

1) $\tau = u$ and $\xi = v$, where $-\infty < u < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of the exponential functions.

2) $\tau = u\cosh v$ and $\xi = u\sinh v$ with $-\infty \le u < \infty$, $-\infty < v < \infty$ yield a product of an exponential and Bessel functions.

3) $\tau = \left(u^2+v^2\right)/2$ and $\xi = uv$ with $-\infty \le u < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of parabolic cylinder functions.

4) $\tau = uv$ and $\xi = \left(u^2+v^2\right)/2$ with $-\infty < v < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of parabolic cylinder functions.

5) $\tau + \xi = 2(u+v)$ and $\tau - \xi = (u-v)^2$ with $-\infty < u$, $v < \infty$ yield a product of Airy functions.

6) $\tau + \xi = \cosh[(u-v)/2]$ and $\tau - \xi = \sinh[(u+v)/2]$ with $-\infty < u$, $v < \infty$ yield a product of Mathieu functions.

7) $\tau + \xi = 2\sinh(u-v)$ and $\tau - \xi = \exp(u-v)$ with $-\infty < u$, $v < \infty$ yield a product of Bessel functions.

8) $\tau + \xi = 2\cosh(u-v)$ and $\tau - \xi = \exp(u+v)$ with $-\infty < u$, $v < \infty$ yield a product of Bessel functions.

9) $\tau = \sinh u \cosh v$ and $\xi = \cosh u \sinh v$ with $-\infty < u$, $v < \infty$ yield a product of Mathieu functions.

10) $\tau = \cosh u \cosh v$ and $\xi = \sinh u \sinh v$ $-\infty < u < \infty$, $-\infty \le v < \infty$ yield a product of Mathieu functions.

11) $\tau = \cos u \cos v$ and $\xi = \sin u \sin v$ $0 < u < 2\pi$, $0 \le v < \pi$ yield a product of Mathieu functions.

The substitutions **1) - 11)** specify some *orthogonal* systems of coordinates (u,v). Besides, there are some *non-orthogonal* systems which enable to separate the variables *u* and *v* as well in the KGE: see paper [14].

References

[1] Eroglu E., Aksoy S., Tretyakov O.A., “Surplus of Energy for Time-Domain Waveguide Modes”, *Energy Edu. Sci. and Tech. Part A: Energy Sci. and Research*, 2012, 29(1), pp. 495-506.

[2] Tretyakov O., A., “Evolutionary Waveguide Equations” Sov. J. Comm. Tech. Electron (English translation of Elektrosvyaz and Radiotekhnika) 1990, 35(2), pp. 7-17.

[3] Tretyakov O. ,A., “Essentials of Nonstationary and Nonlinear Electromagnetic Field Theory, in Analytical and Numerical Methods in Electromagnetic Wave Theory”. Hashimoto M, Idemen M, Tretyakov O., A, Eds., ch. 3. Science House Co. Ltd., Tokyo, 1993.

[4] Tretyakov O., A. Evolutionary equations for the theory of waveguides. IEEE AP-S Int. Symp. Dig. Seattle, pp. 2465-2471, Jun. 1994

[5] Aksoy S and Tretyakov O., A., “Evolution Equations for Analytical Study of Digital Signals in Waveguides”. J. Electromagn. Waves Appl. 2004, 17(12), pp. 263-270.

[6] Aksoy S. and Tretyakov O., A., “Study of A Time Variant Cavity System” Journal of Electromagnetics Waves and Appl., 2002, 16(1), pp. 1535-1553.

[7] Erden F., “Kavitelerde Zorlanmış Salınımların Zaman-Dömeni Analizlerinin Incelenmesi”, Dissertation Thesis, 2009, Gebze Institute of Technology

[8] Tretyakov O., A. and Akgün O., “Derivation of Klein-Gordon Equation from Maxwell’s Equations and Study of Relativistic Time-Domain Waveguide Modes”. Progress In Electromagnetics Research (PIER), 2010, 105, pp. 171-191.

[9] Borisov VV. Transient Electromagnetic Waves. Leningrad Univ. Press, Leningrad, 1987 (in Russian).

[10] Kristensson G. Transient electromagnetic wave propagation in waveguides. J. Electromagn. Waves Appl. 1995, 9(5/6), pp.645-671.

[11] Shvartsburg AB. Single-cycle waveforms and non-periodic waves in dispersive media (exactly solvable models). Phys. Usp. 1998, 41(1), pp. 77-94.

[12] Slivinski A and Heyman E. Time-domain near-field analysis of short-pulse antennas. Part I: Spherical wave (multipole) expansion. IEEE Trans. Anten. Propag. 1999;47:271-279.

[13] Geyi W., “A Time-Domain Theory of Waveguides”. Progress in Electromagnetics Research (PIER) 2006, 59, pp. 267-297.

[14] Miller WJr. Symmetry and Separation of Variables, MA: Addison-Wesley Publication Co., Boston, 1977.