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On Parallel Surfaces of Timelike Ruled Weingarten Surfaces

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ABSTRACT

In this work, it is shown that paralel surfaces of timelike ruled surfaces which are developable are timelike ruled Weingarten surfaces. It is also shown that paralel surfaces of non-developable ruled Weingarten surfaces are again Weingarten surfaces. Finally, some properties of that kind paralel surfaces are obtained in Minkowski 3-space.

1. Introduction

Creation of paralel surfaces is useful in design and manufacture. Making of dies for forging and castings require modeling of paralel surfaces. Enhancing or reducingthe size of free-from surfaces requires calculation of curvature and other properties of the new surface, which is paralel to the original surface.

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite numbers of surfaces because we choose the constant distance along the normal, arbitrarily. From the definition, it follows that

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parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M [26].

A Weingarten (or $W-$) surface is a surface on which there exists a relationship between the principal curvatures. Let f and g be smooth functions on a surface M in Minkowski 3-space. The Jacobi function $\Phi(f,g)$ formed with f, g is defined by $\Phi(f,g)=\det\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ where $f_u=\dfrac{\partial f}{\partial u}$ and $f_v=\dfrac{\partial f}{\partial v}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H)=0$ with respect to the Gaussian curvature K and the mean curvature H is called a Weingarten surface or $W-$ surface. All developable surfaces $(K=0)$ and minimal surfaces $(H=0)$ are Weingarten surfaces. Some geometers have studied Weingarten surfaces and obtained many interesting results in both Euclidean and Minkowskian spaces [2, 3, 4, 7, 8, 9, 10, 14, 15, 16, 24, 25].

In this paper, we study on parallel surfaces of timelike surfaces that become both ruled and Weingarten surfaces. We show that parallel surface of a timelike ruled Weingarten surface is again a Weingarten surface in Minkowski 3-space. Also, some properties of that kind parallel surface are given in Minkowski 3-space.

2. Preliminaries

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space R^3 with the metric

$$<d\mathbf{x},d\mathbf{x}>=dx_1^2+dx_2^2-dx_3^2$$

where (x_1,x_2,x_3) denotes the canonical coordinates in R^3. An arbitrary vector $\mathbf{x}$ of E_1^3 is said to be spacelike if $<\mathbf{x},\mathbf{x}>>0$ or $\mathbf{x}=0$, timelike if $<\mathbf{x},\mathbf{x}><0$ and lightlike or null if $<\mathbf{x},\mathbf{x}>=0$ and $\mathbf{x}\neq 0$. A timelike or light-like vector in E_1^3 is said to be causal. For $x\in E_1^3$, the norm is defined by $\|\mathbf{x}\|=\sqrt{|\langle \mathbf{x},\mathbf{x}\rangle|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $<\mathbf{x},\mathbf{x}>=1$ and a timelike unit vector if $<\mathbf{x},\mathbf{x}>=-1$. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [18]. For any two vectors $\mathbf{x}=(x_1,x_2,x_3)$ and

$\mathbf{y} = (y_1, y_2, y_3)$ of E_1^3, the inner product is the real number $<\mathbf{x}, \mathbf{y}> = x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} >= ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [17].

A surface in the Minkowski 3-space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentzian metric. This is equivalent to saying that the normal vector on the timelike surface is a spacelike vector [1].

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns each $u \in I$ to a point $\alpha(u) \in E_1^3$ and a vector $X(u) \in E_1^3$, $X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiable on u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and parallel to $X(u)$ is called the line of the family at u.

Given one-parameter family of lines $\{\alpha(u), X(u)\}$, the parameterized surface

$$\varphi(u,v) = \alpha(u) + vX(u), \qquad u \in I, \qquad v \in R \tag{1}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$.The lines L_u are called the rulings and the curve $\alpha(u)$ is called a directrix of the surface φ. The normal vector of surface is denoted by N [21]. The coefficients that belong to the parametric equations of the surface given in (1) areas follows;

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle$$

where the differentiable functions $E, F, G : U \to R$ are called the coefficients of the first fundamental form I. So the first fundamental form is

$$I = Edu^2 + 2Fdudv + Gdv^2$$

The following differentiable functions

$$e = -\langle X_u, N_u \rangle = \langle N, X_{uu} \rangle$$

$$f = -\langle X_u, N_v \rangle = -\langle X_v, N_u \rangle = \langle N, X_{uv} \rangle$$

$$g = -\langle X_v, N_v \rangle = \langle N, X_{vv} \rangle$$

are called the coefficients of the second fundamental form II. So the second fundamental form is $II = edu^2 + 2fdudv + gdv^2$ [17].

2.1 Theorem Using standard parameters, a ruled surface in Minkowski 3-space is up to Lorentzian motions, uniquely determined by the following quantities:

$$Q = \langle \alpha',\, X \wedge X' \rangle, \quad J = \langle X,\, X'' \wedge X' \rangle, \quad F = \langle \alpha',\, X \rangle \tag{2}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [16].

2.2 Theorem The Gauss and mean curvatures of timelike ruled surface φ in terms of the parameters Q, J, F, D in E_1^3 are obtained as follows:

i) For timelike ruled surface with spacelike ruling:

$$K = -\frac{Q^2}{D^4} \qquad \text{and}\; H = \frac{1}{2D^3}(-QF + Q^2 J + vQ' + v^2 J) \tag{3}$$

where $D = \sqrt{-\varepsilon Q^2 + \varepsilon v^2}$, $\left\langle X',\, X' \right\rangle = \varepsilon = \pm 1$

ii) For timelike ruled surface with timelike ruling:

$$K = -\frac{Q^2}{D^4} \qquad \text{and} \qquad H = \frac{1}{2D^3}(-QF - Q^2 J + vQ' - Jv^2) \tag{4}$$

where $D = \sqrt{-Q^2 - v^2}$ [7].

2.3 Definition A surface is called a Weingarten surface or $W-$ surface in E_1^3 if there is a nontrivial relation $\Phi(K, H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives with respect to u and v, this is the equation

$$K_u H_v - K_v H_u = 0 \tag{5}$$

where K and H are Gaussian and mean curvatures of surface, respectively [7].

2.4 Theorem The ruled surface φ is a Weingarten surface if and only if the invariants Q, J and F are constant in E_1^3 [7].

2.5 Theorem Parameter curves are lines of curvature if and only if $F = f = 0$ in E_1^3 [17].

2.6 Lemma A point p on a surface M in E_1^3 is an umbilical point if and only if

$$\frac{E}{e}=\frac{F}{f}=\frac{G}{g} \tag{6}$$

[13].

2.7 Definition Let M and M^r be two surfaces in E_1^3. The function

$$\mu : M \to M^r$$

$$p \to \mu(p) = p + r\mathbf{N}_p$$

is called the parallelization function between M and M^r and furthermore M^r is called the parallel surface to M in E_1^3 where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [11].

The representation of points on M^r can be obtained by using the representations of points on M in E_1^3. Let φ be the position vector of a point P on M and φ^r be the position vector of a point $\mu(p)$ on the parallel surface M^r. Then $\mu(p)$ is at a constant distance r from P along the normal to the surface M. Therefore the parameterization for M^r is given by

$$\varphi^r(u,v) = \varphi(u,v) + r\mathbf{N}(u,v) \tag{7}$$

where r is a constant scalar and $\mathbf{N}$ is the unit normal vector field on M [22].

2.8 Theorem Let M be a surface and M^r be a parallel surface of M in E_1^3. Let $\mu : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$,

1. $\mu_*(X) = X + rS(X)$

2. $S^r(\mu_*(X)) = S(X)$

3. μ preserves principal directions of curvature, that is

$$S^r(\mu_*(X)) = \frac{k}{1+rk}\mu_*(X)$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction X [11].

2.9 Theorem Let M be a surface and M^r be a parallel surface of M in E_1^3. Let $\mu : M \to M^r$ be the parallelization function. Then μ preserves becoming umbilical point on the surface M^r in E_1^3 [23].

2.10 Theorem Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K^r = \frac{K}{1+2rH+r^2K} \text{ and } H^r = \frac{H+rK}{1+2rH+r^2K} \tag{8}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r respectively [22].

2.11 Corollary Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K = \frac{K^r}{1-2rH^r+r^2K^r} \text{ and } H = \frac{H^r-rK^r}{1-2rH^r+r^2K^r} \tag{9}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r respectively [22].

2.12 Theorem Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Parallel surface of a timelike developable ruled surface is again a timelike ruled surface [22].

2.13 Theorem Let $\varphi(u,v)$ be a timelike surface in E_1^3 with $F = f = 0$. Then the parallel surface

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

is a developable ruled surface while one of the parameters of parallel surface is constant and the other is variable [22].

2.14 Corollary Let M be a timelike ruled surface and M^r be a timelike parallel surface of M in E_1^3. The Gaussian and mean curvatures K^r and H^r are, respectively, as follows:

i) For timelike ruled surface with spacelike ruling:

$$K^r = \frac{-Q^2}{D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2} \tag{10}$$

$$H^r = \frac{-QFD + Q^2JD + vQ'D + v^2JD - 2rQ^2}{2D^4 - 2rQFD + 2rQ^2JD + 2rvQ'D + 2rv^2JD - 2r^2Q^2} \tag{11}$$

ii) For timelike ruled surface with timelike ruling:

$$K^r = \frac{-Q^2}{D^4 - rQFD - rQ^2JD + rvQ'D - rv^2JD - r^2Q^2} \tag{12}$$

$$H^r = \frac{-QFD - Q^2JD + vQ'D - v^2JD - 2rQ^2}{2D^4 - 2rQFD - 2rQ^2JD + 2rvQ'D - 2rv^2JD - 2r^2Q^2} \tag{13}$$

in terms of the parameters Q, J, F, D [22].

3. Parallel surfaces of timelike ruled Weingarten surfaces

Let M^r be a parallel to a surface M in Minkowski 3-space. If there is a nontrivial relation as

$$\Phi(K^r, H^r) = 0 \tag{14}$$

between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r, the parallel surface M^r is said to be Weingarten surface as in analogous to the original surface. In other words, if Jacobi determinant as a relation between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r vanishes, we have the following condition for parallel Weingarten surfaces

$$\Phi(K^r, H^r) = \det\begin{pmatrix} K_u^r & K_v^r \\ H_u^r & H_v^r \end{pmatrix} = K_u^r H_v^r - K_v^r H_u^r = 0 \tag{15}$$

3.1 Theorem Let M be a developable timelike ruled surface in E_1^3, then the parallel surface M^r of M is a timelike parallel ruled Weingarten surface.

Proof From theorem 2.12, the parallel surface of developable timelike ruled surface M is again a developable timelike ruled surface. Therefore, Gaussian curvature K^r of the parallel surface vanishes since $K=0$ for M. It means that the surface M^r is a Weingarten surface.

3.2 Theorem Let $\varphi(u,v)$ be a timelike surface in E_1^3 with $F=f=0$. Then the parallel surface

$$\varphi^r(u,v)=\varphi(u,v)+r\mathbf{N}(u,v)$$

is a ruled Weingarten surface while one of the parameters of parallel surface is constant and the other is variable.

Proof The surface $\varphi^r(u,v)$ is a developable timelike ruled surface from Theorem 2.13, hence K^r vanishes by putting $K=0$ in theorem 2.11. Consequently, the surface φ^r is also Weingarten surface.

3.1 Parallel surfaces of timelike ruled Weingarten surfaces with spacelike ruling

3.3 Theorem Let φ^r be a parallel surface of a timelike ruled surface φ with spacelike ruling in E_1^3. Then the surface φ is a Weingarten surface if and only if the parallel surface φ^r is a Weingarten surface.

Proof ($\Rightarrow$): Let φ be a timelike surface in E_1^3, then we have to show the equation (15) by using the equation (5). First, using the expressions of (8) in (15), we have

$$\begin{aligned} K_u^r H_v^r - K_v^r H_u^r &= \left(\frac{K}{1+2rH+r^2K}\right)_u \left(\frac{H+rK}{1+2rH+r^2K}\right)_v - \left(\frac{K}{1+2rH+r^2K}\right)_v \left(\frac{H+rK}{1+2rH+r^2K}\right)_u \\ &= \Omega\Big\{\left[K_u+2rK_uH-2rKH_u\right]\left[H_v+K_v+r^2K_vH-r^2KH_v\right]- \\ &\quad \left[K_v+2rK_vH-2rKH_v\right]\left[H_u+rK_u+r^2K_uH-r^2KH_u\right]\Big\} \end{aligned} \tag{16}$$

where $\Omega=\dfrac{1}{1+2rH+r^2K}$. If we make computations in (16), we get

$$\begin{aligned}K_u^r H_v^r - K_v^r H_u^r = \Omega\Big\{ & K_u H_v + rK_u K_v + r^2 K_u K_v H - r^2 K K_u H_v + 2rHK_u H_v + 2r^2 K_u K_v H + \\ & 2r^3 K_u K_v H^2 - 2r^3 KHK_u H_v - 2rKH_u H_v - 2r^2 KK_v H_u - 2r^3 KHK_v H_u + \\ & 2r^3 K^2 H_u H_v - K_v H_u - rK_u K_v - r^2 HK_u K_v + r^2 KK_v H_u - 2rHK_v H_u - \\ & 2r^2 HK_u K_v - 2r^3 H^2 K_u K_v + 2r^3 KHK_v H_u + 2rKH_u H_v + r^2 KK_u H_v + \\ & 2r^3 KHK_u H_v - 2r^3 K^2 H_u H_v \Big\}\end{aligned} \tag{17}$$

After making arrangements in (17), the equation becomes as

$$\begin{aligned}K_u^r H_v^r - K_v^r H_u^r = \frac{1}{1+2rH+r^2K} & \Big\{ \left[K_u H_v - K_v H_u\right] + 2rH\left[K_u H_v - K_v H_u\right] \\ & +2r^3 KH\left[K_u H_v - K_v H_u\right] + r^2 K\left[K_u H_v - K_v H_u\right]\Big\}\end{aligned} \tag{18}$$

Later, using (2.5) in (3.5), we get

$$K_u^r H_v^r - K_v^r H_u^r = 0 \tag{19}$$

Therefore the parallel surface φ^r is a timelike Weingarten surface.

$(\Leftarrow)$: Conversely, let φ^r be a Weingarten surface which is parallel to a timelike ruled surface, then it satisfies (15). First, using the expressions of (9) in (5), we have

$$\begin{aligned}K_u H_v - K_v H_u &= \left(\frac{K^r}{1-2rH^r+r^2K^r}\right)_u \left(\frac{H^r - rK^r}{1-2rH^r+r^2K^r}\right)_v - \left(\frac{K^r}{1-2rH^r+r^2K^r}\right)_v \left(\frac{H^r - rK^r}{1-2rH^r+r^2K^r}\right)_u \\ &= \Gamma\Big\{\left[K_u^r - 2rK_u^r H^r + 2rK_v^r H_u^r\right]\left[H_v^r - rK_v^r + r^2 K_v^r H^r - r^2 K^r H_v^r\right] - \\ &\qquad \left[K_v^r - 2rK_v^r H^r + 2rK^r H_v^r\right]\left[H_u^r - rK_u^r + r^2 K_u^r H^r - r^2 K^r H_u^r\right]\Big\}\end{aligned} \tag{20}$$

where $\Gamma = \dfrac{1}{1-2rH^r+r^2K^r}$. If we make computations in (20), we get

$$\begin{aligned}& K_u H_v - K_v H_u = \\ & \Gamma\Big\{K_u^r H_v^r - rK_u^r H_v^r + r^2 H^r K_u^r H_v^r - r^2 K_u^r H_v^r - 2rH^r K_u^r H_v^r + 2r^2 H^r K_u^r H_v^r - 2r^3 (H^r)^2 K_u^r H_v^r \\ & +2r^3 K^r H^r K_u^r H_v^r + 2rK^r K_u^r H_v^r - 2r^2 K^r K_v^r H_u^r + 2r^3 K^r H^r K_v^r H_u^r - 2r^3 (K^r)^2 K_u^r H_v^r - K_v^r H_u^r \\ & +rK_u^r H_v^r - r^2 H^r K_u^r H_v^r + r^2 K^r K_u^r H_v^r + 2rH^r K_v^r H_u^r - 2r^2 H^r K_u^r K_v^r + 2r^3 (H^r)^2 K_u^r K_v^r \\ & -2r^3 K^r H^r K_v^r H_u^r - 2rK^r H_u^r H_v^r + 2r^2 K^r K_u^r H_v^r - 2r^3 K^r H^r K_u^r H_v^r + 2r^3 (K^r)^2 H_u^r H_v^r\end{aligned} \tag{21}$$

Making arrangements in (21), we get the following equation:

$$\begin{aligned} K_u H_v - K_v H_u &= \left(\frac{K^r}{1-2rH^r+r^2K^r}\right)_u \left(\frac{H^r-rK^r}{1-2rH^r+r^2K^r}\right)_v - \left(\frac{K^r}{1-2rH^r+r^2K^r}\right)_v \left(\frac{H^r-rK^r}{1-2rH^r+r^2K^r}\right)_u \\ &= \Gamma\Big\{\left[K^r_u - 2rK^r_u H^r + 2rK^r_v H^r_u\right]\left[H^r_v - rK^r_v + r^2K^r_v H^r - r^2K^r H^r_v\right] - \\ &\quad \left[K^r_v - 2rK^r_v H^r + 2rK^r H^r_v\right]\left[H^r_u - rK^r_u + r^2K^r_u H^r - r^2K^r H^r_u\right]\Big\} \end{aligned} \tag{22}$$

By using (15) in (22), we have

$$K_u H_v - K_v H_u = 0 \tag{23}$$

Therefore timelike ruled surface φ is a Weingarten surface.

3.4 Corollary The surface φ^r which is parallel to timelike ruled surface φ with spacelike ruling is a Weingarten surface if and only if the invariants Q, J, F which determine timelike ruled surface φ with spacelike ruling are constants.

Proof (⇒): Let the parallel surface φ^r be a Weingarten surface. Then from Theorem 3.3, the timelike ruled surface φ with spacelike ruling is also a Weingarten surface. Hence, the invariants Q, J, F are constants from Theorem 2.4.

(⇐): Let the invariants Q, J, F be constants, then the timelike ruled surface φ with spacelike ruling is a Weingarten surface from Theorem 2.4. The parallel surface φ^r is also a Weingarten surface from Theorem 3.3.

3.5 Corollary Let the surfaces φ and φ^r be timelike ruled Weingarten surface with spacelike ruling and its parallel surface in E^3_1, respectively. Then, there is the relation

$$H^r = \left(r + \frac{H}{K}\right)K^r \tag{24}$$

among Gauss K and mean H curvatures of the timelike ruled Weingarten surface φ with spacelike ruling and Gauss K^r and mean H^r curvatures of the timelike parallel Weingarten surface φ^r.

Proof We will use the values of the curvatures K^r and H^r given in (10) and (11). Let

$$A = D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2 \tag{25}$$

By using (25) in (10), we get

$$A = -\frac{Q^2}{K^r} \tag{26}$$

or

$$A = \frac{-QFD + Q^2JD + vQ'D + v^2JD - 2rQ^2}{2H^r} \tag{27}$$

From (26) and (27), we have

$$2H^rQ^2 = \left[(QF - Q^2J - vQ' - v^2J)D + 2rQ^2\right]K^r \tag{28}$$

By using the expressions in (i) of Theorem 2.2 in (28), we have

$$H^r = \frac{-2D^4H + 2rQ^2}{2Q^2}K^r \tag{29}$$

By using the expressions in theorem 2.2 (i) in (29), we get $H^r = \left(r + \dfrac{H}{K}\right)K^r$.

3.6 Lemma Let φ be a non-developable timelike ruled surface with spacelike ruling and φ^r be parallel surface of φ in E_1^3. Then there is no umbilical point on the timelike parallel Weingarten surface φ^r.

Proof Let the timelike ruled Weingarten surface with spacelike ruling φ be given as the following parameterization:

$$\varphi(u,v) = \alpha(u) + vX(u), \quad \langle\alpha',\alpha'\rangle = -1, \quad \langle X,X\rangle = 1 \text{ and } \langle X',X'\rangle = -1 \tag{30}$$

From theorem 2.4, the invariants Q, J, F have to be constants for ruled surface to become Weingarten surface. If there is an umbilical point on timelike ruled Weingarten surface with spacelike ruling, from lemma 2.6, it has to be

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g} \tag{31}$$

The coefficients of the first fundamental form I for the surface φ are as follows:

$$E = \langle\varphi_u,\varphi_u\rangle = -1 - v^2,\ F = \langle\varphi_u,\varphi_v\rangle = \langle\alpha',X'\rangle,\ G = \langle\varphi_v,\varphi_v\rangle = 1 \tag{32}$$

Thereby the normal vector $\mathbf{N}$ of the surface φ is $\mathbf{N} = \alpha' \wedge X + vX' \wedge X$. The coefficients of the second fundamental form II for the surface φ are obtained as

$$e = \langle \phi_{uu}, \mathbf{N} \rangle = \langle \alpha'', \alpha' \wedge X \rangle + v\langle \alpha'', X' \wedge X \rangle + v\langle X'', \alpha' \wedge X \rangle + v^2 \langle X'', X' \wedge X \rangle$$

$$f = \langle \phi_{uv}, \mathbf{N} \rangle = \langle \alpha', X \wedge X' \rangle \tag{33}$$

$$g = \langle \phi_{vv}, \mathbf{N} \rangle = 0 \tag{34}$$

By using (32), (33) and (34) in (6), we have

$$Fg - Gf = -\langle X, X \rangle \langle \alpha', X \wedge X' \rangle \tag{35}$$

The equation (35) means that there is no umbilical point on a non-developable timelike ruled surface φ with spacelike ruling. Hence there is also no umbilical point on the parallel surface φ^r of φ from theorem 2.9.

3.2 Parallel surfaces of timelike ruled Weingarten surfaces with timelike ruling

3.7 Theorem Let φ^r be a parallel surface of a timelike ruled surface φ with timelike ruling in E_1^3. If φ is a Weingarten surface if and only if φ^r is a Weingarten surface.

Proof This theorem can be proved as similar to the proof of Theorem 3.3.

3.8 Corollary The surface φ^r which is parallel to the timelike ruled surface φ with timelike ruling is a Weingarten surface if and only if the invariants Q, J, F which determine the timelike ruled surface φ with timelike ruling are constants.

Proof (⇒) Let the parallel surface φ^r be a Weingarten surface. Then from theorem 3.7, the timelike ruled surface φ with timelike ruling is also a Weingarten surface. Hence, the invariants Q, J, F are constants from theorem 2.4.

$(\Leftarrow)$: Let the invariants Q, J, F be constants, then the timelike ruled surface φ with timelike ruling is a Weingarten surface from theorem 2.4. The parallel surface φ^r is also a Weingarten surface from theorem 3.7.

3.9 Corollary Let the surfaces φ and φ^r be timelike ruled Weingarten surface with timelike ruling and its parallel surface in E_1^3, respectively.

Then, there is the relation

$$H^r = \left(r + \frac{H}{K}\right)K^r \tag{36}$$

among Gauss K and mean H curvatures of the timelike ruled Weingarten surface φ with timelike ruling and Gauss K^r and mean H^r curvatures of the timelike parallel Weingarten surface φ^r.

Proof We will use the values of the curvatures K^r and H^r given in (10) and (11). Let

$$A = D^4 - rQFD - rQ^2JD + rvQ'D - rv^2JD - r^2Q^2 \tag{37}$$

By using (37) in (10), we get

$$A = -\frac{Q^2}{K^r} \tag{38}$$

or

$$A = \frac{QFD - Q^2JD - vQ'D - v^2JD + 2rQ^2}{2H^r} \tag{39}$$

From (38) and (39), we have

$$2H^rQ^2 = \left[\left(QF + Q^2J - vQ' + v^2J\right)D + 2rQ^2\right]K^r \tag{40}$$

By using the expressions in theorem 2.2 (ii) in (40), we have

$$H^r = \frac{-2D^4H + 2rQ^2}{2Q^2}K^r \tag{41}$$

By using the expressions in theorem 2.2 (ii) in (41), we get $H^r = \left(r + \frac{H}{K}\right)K^r$

3.10 Lemma Let be a non-developable timelike ruled surface with timelike ruling and φ^r be parallel surface of φ in E_1^3. Then there is no umbilical point on the timelike parallel Weingarten surface φ^r.

Proof Let the timelike ruled Weingarten surface with timelike ruling φ be given as the following parameterization:

$$\varphi(u,v) = \alpha(u) + vX(u), \quad \langle \alpha', \alpha' \rangle = 1, \quad \langle X, X \rangle = -1 \text{ and } \langle X', X' \rangle = 1 \tag{42}$$

From theorem 2.4, the invariants Q, J, F have to be constants for ruled surface to become Weingarten surface. If there is an umbilical point on timelike ruled Weingarten surface with timelike ruling, from lemma 2.6, it has to be

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g} \tag{43}$$

The coefficients of the first fundamental form I for the surface φ are as follows:

$$E = \langle \varphi_u, \varphi_u \rangle = 1 + v^2, \; F = \langle \varphi_u, \varphi_v \rangle = \langle \alpha', X' \rangle, \; G = \langle \varphi_v, \varphi_v \rangle = -1 \tag{44}$$

Thereby the normal vector $\mathbf{N}$ of the surface φ is $\mathbf{N} = a' \wedge X + vX' \wedge X$. The coefficients of the second fundamental form II for the surface φ are obtained as

$$e = \langle \phi_{uu}, \mathbf{N} \rangle = \langle \alpha'', \alpha' \wedge X \rangle + v\langle \alpha'', X' \wedge X \rangle + v\langle X'', \alpha' \wedge X \rangle + v^2 \langle X'', X' \wedge X \rangle$$

$$f = \langle \phi_{uv}, \mathbf{N} \rangle = \langle \alpha', X \wedge X' \rangle \tag{45}$$

$$g = \langle \phi_{vv}, \mathbf{N} \rangle = 0 \tag{46}$$

By using (44), (45) and (46) in (6), we have

$$Fg - Gf = -\langle X, X \rangle \langle \alpha', X \wedge X' \rangle \tag{47}$$

The equation (47) means that there is no umbilical point on a non-developable timelike ruled surface φ with timelike ruling. Hence there is also no umbilical point on the parallel surface φ^r of φ from Theorem 2.9.

3.11 Example The third kind of the helicoidal surface can be given with the parameterization:

$$\varphi(u,v) = (v \cosh u, \; v \sinh u, \; au)$$

The coefficients of the first fundamental form I are $E = v^2 + 1, \; F = 0, \; G = -1$. This surface is entirely a timelike ruled surface since $\det I = EG - F^2 = -1^2 - v^2 < 0$. If we compute the values of the invariants Q, J, F in Theorem2.1 for that surface, then we get

$$Q = -1,\ J = 0 \text{ and } F = 0 \tag{48}$$

The invariants Q, J, F in (48) are constant therefore, from corollary 3.4, parallel surface φ^r is a Weingarten surface. The graph figures out the third kind of helicoidal timelike ruled surface, see Figure 1. The unit normal vector is $\mathbf{N} = \frac{1}{\sqrt{1+v^2}}(\sinh u, \cosh u, -v)$ and the coefficients of the second fundamental form II are obtained as $e = 0, f = \frac{1}{\sqrt{1+v^2}}, g = 0$. The parallel surface of the third kind of timelike helicoidal ruled surface is parameterized as follows:

$$\varphi^r(u,v) = \left(v\cosh u + \frac{r\sinh u}{\sqrt{1+v^2}}, v\sinh u + \frac{r\cosh u}{\sqrt{1+v^2}}, u - \frac{rv}{\sqrt{1+v^2}} \right) \tag{49}$$

The Gaussian and mean curvatures of this parallel surface are found as follows:

$$K^r = \frac{1}{(1+v^2)+r^2} \text{ and } H^r = \frac{2r}{(1+v^2)+r^2} \tag{50}$$

The parallel surface of timelike helicoidal ruled surface is also a Weingarten surface since the partial derivatives of K^r and H^r in terms of the variable u vanish. The graph figures out the third kind of helicoidal timelike ruled surface (red one) and its parallel surface (blue one), see Figure 2.

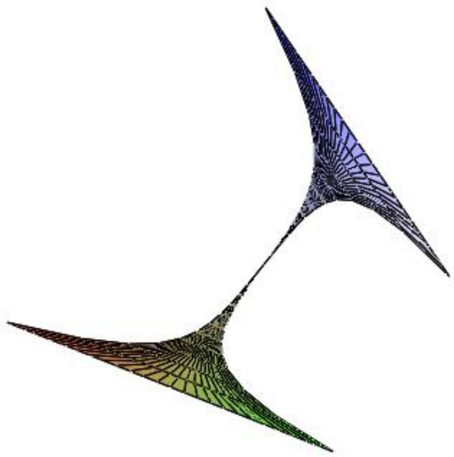

Figure 1. The timelike surface

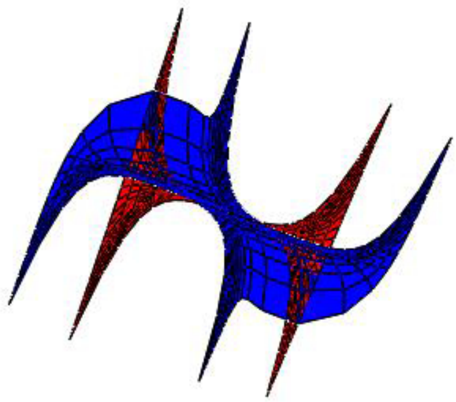

Figure 2. Parallel surface

3.12 Example The surface called as two dimensional ivor is parameterized as:

$\varphi(u,v) = (v\cosh u, v\sinh u, u+v)$

[19].

The coefficients of the first fundamental form I are $E = 1+v^2$, $F = 1$, $G = 0$. This surface is entirely a timelike surface since $\det I = EG - F^2 = -1 < 0$. If we compute the values of the invariants Q, J, F in Theorem 2.1 for that surface, then we get

$$Q = -1,\ J = 1 \text{ and } F = 0. \tag{51}$$

The invariants Q, J, F in (51) are constant therefore, from corollary 3.4, parallel surface φ^r is a Weingarten surface. The graph figures out the two dimensional ivor, see Figure 3.

The unit normal vector is $\mathbf{N} = (\sinh u - v\cosh u, \cosh u - v\sinh u, -v)$. The coefficients of the second fundamental form II are found as $e = v^2$, $f = 1$, $g = 0$.

The parallel surface of the two dimensional ivor is parameterized as follows:

$$\varphi^r(u,v) = (v\cosh u + r\sinh u - vr\cosh u, v\sinh u + r\cosh u - vr\sinh u, u + v - rv) \tag{52}$$

The Gaussian and mean curvatures of this parallel surface are found as follows:

$$K^r = \frac{1}{(1+r)^2} \text{ and } \qquad H^r = \frac{1}{1+r} \tag{53}$$

The parallel surface of the two dimensional ivor is also a Weingarten surface since the partial derivative of K^r and H^r in terms of the variable u and v vanish. The graph figures out the two dimensional ivor (red one) and its parallel surface (blue one), see Figure 4.

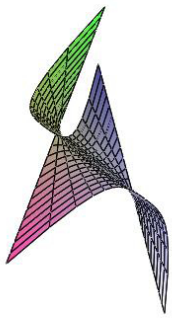

Figure 3. The timelike surface

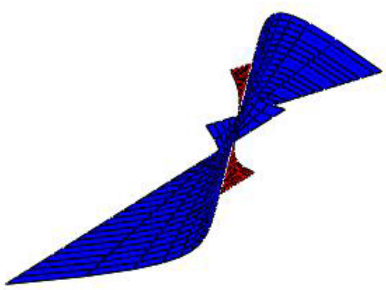

Figure 4. Paralel surface

4. Conclusion

In this study, parallel surface of timelike surface has been taken into consideration as both ruled and Weingarten surface. The parallel surface of a ruled Weingarten surface has been shown to become again a Weingarten surface in Minkowski 3-space. Also, some properties of that kind parallel surface have been given in Minkowski 3-space.

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