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Spacelike Conchoid Curves in the Minkowski Plane

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ABSTRACT

A conchoid $d(t)$ is a curve derived from a fixed point O, another curve $c(t)$, and a constant length d. For every line through O that intersects the given curve at A the two points on the line which are d distance from A are on the conchoid. In this paper, we studied spacelike conchoid curves in the Minkowski plane.

1. Introduction

Let $\mathbf{L}^2$ be a Minkowski plane with the Lorentzian inner product $<\ ,\ >$ given by

$$<\mathbf{x},\mathbf{y}> = x_1 y_1 - x_2 y_2$$

where $\mathbf{x} = \left(x_1, x_2\right)$ and $\mathbf{y} = \left(y_1, y_2\right) \in \mathbf{L}^2$.

A vector $\mathbf{x}$ is said to be timelike if $<\mathbf{x},\mathbf{x}> < 0$, spacelike if $<\mathbf{x},\mathbf{x}> > 0$ or $\mathbf{x} = 0$, and lightlike (or null) if $<\mathbf{x},\mathbf{x}> = 0$ and $\mathbf{x} \neq 0$. The norm of $\mathbf{x}$ is defined by

$$\left\|\mathbf{x}\right\| = \sqrt{\left|<\mathbf{x},\ \mathbf{x}>\right|} \tag{1}$$

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An arbitrary curve $c(t)$ in $\mathbf{L}^2$ can locally be spacelike, timelike or null, if all of its velocity vectors $\frac{dc(t)}{dt}$ are spacelike, timelike or null, respectively. More information about both the Minkowski plane and space can be found in [2, 3, 4, 7].

According to Yaglom [4], the circles with radius r centered at the origin are given by

$$x^2 - y^2 = \pm r^2 \tag{2}$$

We say hat the circles are the first or second kind according as its radius is $+r^2$ or $-r^2$ in the Minkowski geometry.

Map of the complete (x, y) plane by hyperbolic polar transformation is shown in Table 1 [3].

Table 1.

$\lvert x\rvert > \lvert y\rvert$		$\lvert y\rvert > \lvert x\rvert$	
Right Sector	**Left Sector**	**Right Sector**	**Left Sector**
$x = r\cosh t$	$x = -r\cosh t$	$x = r\sinh t$	$x = -r\sinh t$
$y = r\sinh t$	$y = -r\sinh t$	$y = r\cosh t$	$y = -r\cosh t$

where $r = \sqrt{\lvert x^2 - y^2\rvert}$.

For an analytic representation of conchoid curve in the Euclidean plane, let us choose $O = (0,0)$. Using a representation of a curve in terms of polar coordinates $c(t) = r(t)(\cos t, \sin t)$, its conchoid curve $d(t)$ with respect to O and distance d is obtained by

$$c(t) = r(t)k(t), \qquad d(t) = (r(t) \pm d)k(t) \tag{3}$$

where $\lVert \mathbf{k} \rVert = 1$ [1, 6].

Since there are two kind circles in Minkowski plane, we will distinguish following two kind of conchoid curves.

2. Spacelike Conchoid Curve of Type I

Let $k(t)$ e a parameterization of the unit circle of first kind in the Minkowski plane. Let $c(t)$ be a curve in terms of hyperbolic polar coordinates, its conchoid curve D with respect to O and distance d given by

$$c(t) = r(t)k(t), \qquad d(t) = (r(t) \pm d)k(t) \tag{4}$$

where $\|\mathbf{k}\| = 1$.

Note that $\{\mathbf{k}, \mathbf{k}'\}$ are an orthonormal basis in $\mathbf{L}^2$. The vectors $\mathbf{k}$ and $\mathbf{k}'$ are spacelike and timelike, respectively.

Differentiating (4) with respect to t then, we have

$$d' = r' \mathbf{k} + (r + d) \mathbf{k}' \tag{5}$$

From (1) we obtain

$$\|d'\| = \sqrt{\left| r'^2 - (r + d)^2 \right|} \tag{6}$$

For $r'^2 > (r+d)^2$ we have spacelike conchoid curve of type I in Minkowski plane. Spacelike tangent vector t of concoid curve of type I is obtained by

$$\mathbf{t} = \frac{r'\mathbf{k} + (r+d)\mathbf{k}'}{\sqrt{r'^2 - (r+d)^2}} \tag{7}$$

The timelike normal vector is

$$\mathbf{n} = \frac{r'\mathbf{k}' + (r+d)\mathbf{k}}{\sqrt{r'^2 - (r+d)^2}} \tag{8}$$

We may express the relationship between the frames $\{\mathbf{k}, \mathbf{k}'\}$ and $\{\mathbf{t}, \mathbf{n}\}$ in matrix form as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \end{bmatrix} = \frac{1}{\sqrt{r'^2 - (r+d)^2}} \begin{bmatrix} r' & r+d \\ r+d & r' \end{bmatrix} \begin{bmatrix} \mathbf{k} \\ \mathbf{k}' \end{bmatrix} \tag{9}$$

Differentiating (5) with respect to t then, using $\mathbf{k}'' = \mathbf{k}$, we obtain

$$d'' = (r'' + r + d)\mathbf{k} + 2r'\mathbf{k}' \tag{10}$$

Substituting (10) and (5) into $\kappa = \det(d', d'') / \|d'\|^3$ the curvature of conchoid curve of type I obtained by

$$\kappa = \frac{2r'^2 - (r'' + r + d)(r + d)}{\left(\sqrt{r'^2 - (r+d)^2}\right)^3} \tag{11}$$

On the other hand, we can define the hyperbolic angle ψ between the timelike vectors $\mathbf{k}'$ and $\mathbf{n}$, then we have

$$< \mathbf{k}', \mathbf{n} >= \cosh\psi \tag{12}$$

From (7) and (8) we have

$$\cosh\psi = -\frac{r'}{\sqrt{r'^2 - (r+d)^2}} \tag{13}$$

and

$$\sinh\psi = -\frac{(r+d)}{\sqrt{r'^2 - (r+d)^2}} \tag{14}$$

3. Spacelike Conchoid Curve of Type II

Let $k_1(t)$ be a parameterization of the unit circle second kind in the Minkowski plane. Let $c_1(t) = r(t)k_1(t)$ be a curve in terms of hyperbolic polar coordinates, its conchoid curve $d(t)$ with respect to O and distance d is obtained by

$$d_1(t) = (r(t) \pm d)k_1(t) \tag{15}$$

We have an orthonormal basis $\{\mathbf{k}_1, \mathbf{k}_1'\}$ and $\{\mathbf{t}_1, \mathbf{n}_1\}$ in $\mathbf{L}^2$, where $\mathbf{k}_1$ and $\mathbf{k}_1'$ are timelike and spacelike vectors, respectively.

Differentiating (15) with respect to t then, from (1) we obtain

$$\|d'\| = \sqrt{(r+d)^2 - r'^2} \tag{16}$$

For $(r+d)^2 > r'^2$ we have spacelike conchoid curve of type II in Minkowski plane. We may express the relationship between the frames $\{\mathbf{k}_1, \mathbf{k}_1'\}$ and $\{\mathbf{t}_1, \mathbf{n}_1\}$ in matrix form as

$$\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \end{bmatrix} = \frac{1}{\sqrt{r'^2 - (r+d)^2}} \begin{bmatrix} r' & r+d \\ r+d & r' \end{bmatrix} \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{k}_1' \end{bmatrix} \tag{17}$$

The curvature κ of conchoid curve of type II can be obtained by analogous way as in the equation (11).

3.1 Example Let $y=1$ be a line whose conchoid curve of type I and type II we want to construct in the Minkowski plane. If $|x|>1$ then using equations for right sector, shown in Table 1, the hyperbolic polar transformation of the line can be written in the following form:

$$\frac{1}{\sinh t} = r(t) \tag{18}$$

Using (4) and (18) implies that

$$c(t) = \frac{1}{\sinh t}(\cosh t, \sinh t) \tag{19}$$

Hence, conchoid curve of type I is obtained by

$$d(t) = \left(\frac{1 + d\sinh t}{\sinh t}\right)(\cosh t, \sinh t) \tag{20}$$

or from equations for left sector in Table 1, we have another representation of conchoid curve of type I as

$$d(t) = \left(\frac{d\sinh t - 1}{\sinh t}\right)(-\cosh t, -\sinh t) \tag{21}$$

By eliminating the parameter t in the above equation, the conchoid curve of type I is obtained as an implicit algebraic equation of x and y in the following form:

$$x^2(y-1)^2 - y^2(y-1)^2 - y^2d^2 = 0 \tag{22}$$

In an analogous way if $\left|x\right| < 1$ then, conchoid curve of type II is represented by

$$d_1(t) = \left(\frac{1 + d\cosh t}{\cosh t}\right)(\sinh t, \cosh t) \tag{23}$$

$$d_1(t) = \left(\frac{-1 + d\cosh t}{\cosh t}\right)(-\sinh t, -\cosh t) \tag{24}$$

By eliminating the parameter t we find that

$$y^2(y-1)^2 - x^2(y-1)^2 - y^2d^2 = 0 \tag{25}$$

Consequently, Figure 1a and 1b show results obtained from the conchoid transformation of curve in the Euclidean and Minkowski planes.

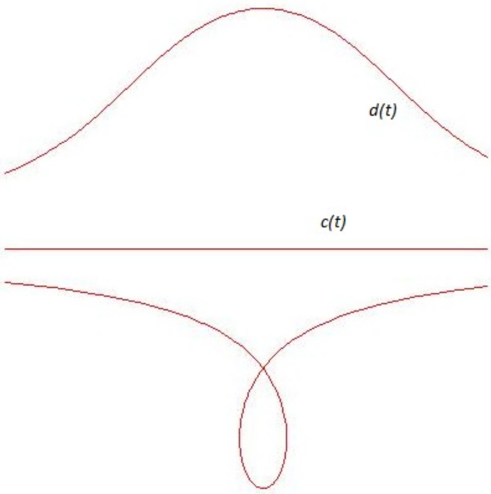

Figure 1a. The conchoid of $y = 1$ line in the Euclidean plane.

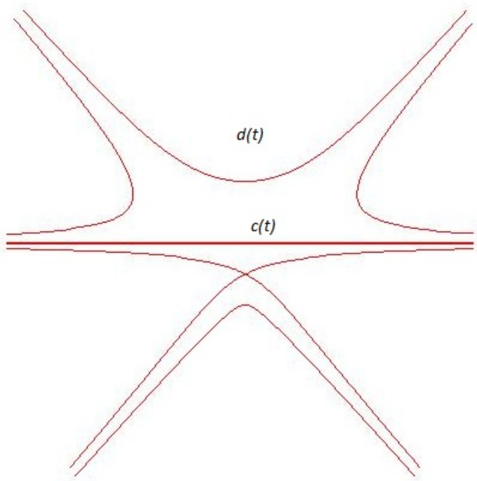

Figure 1b. The conchoid of $y = 1$ line in the Minkowski plane.

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