BALKANJM 02 (2014) 163-172

Contents lists available at BALKANJM

BALKAN JOURNAL OF MATHEMATICS

journal homepage: www.balkanjm.com

The Solutions of Four Riccati Difference Equations Associated with Fibonacci Numbers

D.Turgut Tollu[a], Yasin Yazlik[*b], Necati Taskara[c]

[a] *Necmettin Erbakan University, Department of Mathematics-Computer Sciences, Konya, Turkey*

[b] *Nevsehir Haci Bektas Veli University, Department of Mathematics, Nevsehir, Turkey*

[c] *Selcuk University, Department of Mathematics, Konya, Turkey*

ARTICLE INFO

Article history:
Received 30.12.2013
Accepted 04.03.2014
Available online 30.05.2014

Keywords:
Riccati difference equation,
Fibonacci numbers, golden ratio,
forbidden set.

ABSTRACT

In this paper, we consider the following difference equations

$$x_{n+1}=\frac{1+x_n}{x_n},\ y_{n+1}=\frac{1-y_n}{y_n},\ u_{n+1}=\frac{1}{u_n+1},\ v_{n+1}=\frac{1}{v_n-1}$$

with real initial values x_0, y_0, u_0, v_0 such that the denominators always are nonzero for $n\in\mathbb{N}_0$. We show that these equations is closely linked with each other and also well-known Fibonacci numbers. Here we only study the first equation and apply obtained results to the others.

1. Introduction

Recently, there have been many researches and interest in the field of the theories of dynamical systems and difference equations by several authors [1-15]. Most of the recent applications of these theories have appeared in many scientific areas such as biology, economics, physics, resource management. Especially, nonlinear difference equations of order one have great importance in applications. Also, there are studies that these equations appear as discrete analogues and numerical solutions of differential equations modeling some problems in some branches of science. Probably the most typical example of such equations is Riccati difference equation

[*] Corresponding author: *E-mail*: yyazlik@nevsehir.edu.tr (Y. Yazlik).
2014.002.03

$$x_{n+1}=\frac{a+bx_n}{c+dx_n},\ n\in\mathbb{N}_0$$

with initial value x_0, which has a great variety of properties and which is often studied in textbooks and the literature, see, for example, book [1] and [4,9,13].

We now present some studies in the literature. In [4], Papaschinopouls and Papadopoulos studied on the fuzzy difference equation

$$x_{n+1}=A+\frac{B}{x_n}$$

which is a special case of Riccati difference equation. Cinar, in [5], posed the difference equation

$$x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}}$$

can be transformed into a Riccati difference equation, and got the formulae for solution of the equation. Also, some extensions of the above equation were overworked by many researchers, i.e see [6-8].

In addition, the studies about the relation between Fibonacci numbers and difference equations or systems have been received great attention by researchers, recently. For example; in [11], Elabbasy et al. obtained Fibonacci sequence in solutions of some special cases of following difference equation

$$x_{n+1}=\frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cx_{n-q}}$$

In [12], Elsayed dealt with behavior of solution of the nonlinear difference equation

$$x_{n+1}=ax_{n-1}+\frac{bx_nx_{n-1}}{cx_n+dx_{n-2}}$$

Also, he gave specific forms of the solutions of four special cases of this equation. These specific forms also contain Fibonacci numbers. In [10], Simsek et al. studied the behavior of the solutions of the following system of difference equations

$$x_{n+1}=\max\left\{\frac{A}{x_n},\frac{y_n}{x_n}\right\},\ \ y_{n+1}=\max\left\{\frac{A}{y_n},\frac{x_n}{y_n}\right\}$$

Further, they established a relationship between Fibonacci numbers and solutions of system.

By the motivation of [10-13], our aim in this paper is to explain that the following Riccati difference equations

$$x_{n+1}=\frac{1+x_n}{x_n},\quad n\in\mathbb{N}_0 \tag{1.1}$$

$$y_{n+1}=\frac{1-y_n}{y_n},\quad n\in\mathbb{N}_0 \tag{1.2}$$

$$u_{n+1}=\frac{1}{u_n+1},\quad n\in\mathbb{N}_0 \tag{1.3}$$

$$v_{n+1}=\frac{1}{v_n-1},\quad n\in\mathbb{N}_0 \tag{1.4}$$

where initial values $x_0,\ y_0,\ u_0,\ v_0$ are real numbers such that the denominators always are nonzero, are closely linked with each other and to show that there are some relationships between Fibonacci numbers and solutions of these equations. Here we only study equation (1.1) and apply obtained results to the others. Note that equation (1.3) and (1.4) was studied in [13]. But, in this paper, we again deal with these equations in a different way and more demonstratively prove our results. The above mentioned four equations can be transformed into each other's via the following changes of variables

$$x_n=-y_n=\frac{1}{u_n}=-\frac{1}{v_n},\quad n\in\mathbb{N}_0 \tag{1.5}$$

This property provides a basis for our study.

2. Preliminaries

Here we will give some information about Fibonacci numbers and the golden ratio that naturally rise in our study. Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ are defined by

$$F_{n+2}=F_{n+1}+F_n,\quad n\in\mathbb{N}_0 \tag{2.1}$$

where $F_0=0,\ F_1=1.$ Equation (1.3) has the characteristic equation

$$\lambda^2-\lambda-1=0 \tag{2.2}$$

with $\lambda_+=\frac{1+\sqrt{5}}{2}=\alpha$ and $\lambda_-=\frac{1-\sqrt{5}}{2}=\beta$. The following formula which describes Fibonacci numbers

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad n\in\mathbb{N}_0 \tag{2.3}$$

is called Binet formula of Fibonacci numbers. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n}=F_{-n+2}-F_{-n+1}=(-1)^{n+1}F_n \tag{2.4}$$

Fibonacci numbers have been interested by the researchers for a long time to get main theory and applications of these numbers. For related studies, see, for example [16-21]. The ratio of two consecutive Fibonacci numbers converges to the number $\alpha = \frac{1+\sqrt{5}}{2}$ called the golden ratio. More generally, we can give the following limit

$$\lim_{n\to\infty} \frac{F_{n+r}}{F_n} = \alpha^r, \ r \in \mathbb{Z} \tag{2.5}$$

The applications of the golden ratio appear in many research areas, particularly in physics, engineering, architecture, nature and art. Physicists Naschie and Marek-Crnjac gave some examples of the Golden ratio in theoretical physics and physics of high energy particles[18,19]. For further detail related to Fibonacci numbers and the golden ratio, see [21].

Let us consider the following lemma which will be needed for the results in this study.

Lemma 2.1 [21] For $n \geq 0$, $\alpha^n = \alpha F_n + F_{n-1}$ and $\beta^n = \beta F_n + F_{n-1}$.

Before coming main results we introduce a notion called forbidden set. This set contain all the initial values which produce the undefined solutions of the considered equation.

Let

$$x_{n+1} = f\left(x_n\right), \ n \in \mathbb{N}_0 \tag{2.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function, be a difference equation of order one and D be the domain of the function *f*. Forbidden set of difference equation (2.6) is given by

$$\mathcal{F} = \left\{x_0 \in \mathbb{R} : x_i \in D \ \mathit{for} \ i \in \{0,1,\ldots,n\}, \ n \in \mathbb{N}_0, \ \mathit{and} \ x_{n+1} \notin D\right\}$$

That is, if the initial values are chosen from the complement of the forbidden set then the solution will always become well-defined.

3. Main Results

In this section that we prove the main results, we derive a formula for the solution of equation (1.1) and realize that this solution is connected with well-known Fibonacci numbers. Previously, let us consider the following lemma.

Lemma 3.1 Forbidden sets of initial values x_0, y_0, u_0 and v_0 for equations (1.1)-(1.4) is given by

$$\mathcal{F}_1 = \left\{ x_0 \in \mathbb{R} : x_0 = -\frac{F_n}{F_{n+1}}, \ \mathit{for} \ n \in \mathbb{N}_0 \right\},$$

$$\mathcal{F}_2 = \left\{ y_0 \in \mathbb{R} : y_0 = \frac{F_n}{F_{n+1}}, \ \mathit{for} \ n \in \mathbb{N}_0 \right\},$$

$$\mathcal{F}_3 = \left\{ u_0 \in \mathbb{R} : u_0 = -\frac{F_{n+1}}{F_n}, \ \mathit{for} \ n \in \mathbb{N}_0 \right\},$$

and

$$\mathcal{F}_4 = \left\{ v_0 \in \mathbb{R} : v_0 = \frac{F_{n+1}}{F_n},\ for\ n \in \mathbb{N}_0 \right\},$$

where F_n is nth Fibonacci number, respectively.

Proof. We will prove for equation (1.1) and will apply the obtained result to equations (1.2)-(1.4). Forbidden set is the set of all the points in the complement of domain of function associated with a difference equation. Now, we will determine the set of these points. So, let $f(x) = \frac{1+x}{x}$ be function associated with equation (1.1). Note that *f* is not defined on the point $x = 0$. According to the definition, the string of points $x_0, x_1, \ldots, x_{n_0} = 0,\ n_0 \in \mathbb{N}_0,$ can be determined but x_{n_0+1} cannot be determined. Hence we can write

$$x_{n_0} = f^{n_0}(x_0) = 0,$$

which implies

$$f^{-n_0}(0) = x_0. \tag{3.1}$$

Inverse of function *f* can be easily calculated as $f^{-1}(x) = \frac{1}{x-1}$. It is interesting that the difference equation associated with inverse function f^{-1} is equation (1.4) whose solution was obtained as

$$v_n = -\frac{F_n - F_{n-1}v_0}{F_{n+1} - F_n v_0} = f^{-n}(v_0),\ n \in \mathbb{N}_0, \tag{3.2}$$

in [13]. Consequently, by using solution (3.2), from (3.1), we have

$$x_0 = f^{-n_0}(0) = -\frac{F_{n_0}}{F_{n_0+1}}.$$

It means that if the initial value x_0 is chosen from the forbidden set $\mathcal{F}_1$ then the term x_{n_0+1} cannot be calculated. From (1.5), by some computation, the results for equations (1.2)-(1.4) immediately follow.□

Lemma 3.1 provides the solutions that we get in the following theorem are well-defined. Now we can safely state the theorem.

Theorem 3.2 Let the initial values be $x_0 \in \mathbb{R} - \mathcal{F}_1,\ y_0 \in \mathbb{R} - \mathcal{F}_2,\ u_0 \in \mathbb{R} - \mathcal{F}_3,\ v_0 \in \mathbb{R} - \mathcal{F}_4.$ Then, the formulae for the solutions of equations (1.1)-(1.4) are given by

$$x_n = \frac{F_{n+1}x_0 + F_n}{F_n x_0 + F_{n-1}}, \tag{3.3}$$

$$y_n = \frac{F_{-(n+1)}y_0 + F_{-n}}{F_{-n}y_0 + F_{-(n-1)}}, \tag{3.4}$$

$$u_n = \frac{F_n + F_{n-1}u_0}{F_{n+1} + F_n u_0}, \tag{3.5}$$

$$v_n = \frac{F_{-n} + F_{-(n-1)}v_0}{F_{-(n+1)} + F_{-n}v_0}, \tag{3.6}$$

where F_n is nth Fibonacci number, F_{-n} is nth negative Fibonacci number and $n \in \mathbb{N}_0$.

Proof. We apply the change of variables

$$x_n = \frac{w_n}{w_{n-1}} \tag{3.7}$$

to equation (1.1). In this case, equation (1.1) transformed into the following linear difference equation of order two

$$w_{n+1} = w_n + w_{n-1}. \tag{3.8}$$

We can write equation (3.8) in the operational form

$$(E-\alpha)(E-\beta)w_{n-1} = 0, \tag{3.9}$$

where α and β are the roots of equation (2.2). Now, from (3.9), we have

$$w_{n+1} - \beta w_n = \alpha(w_n - \beta w_{n-1}), \tag{3.10}$$

which yields

$$w_n - \beta w_{n-1} = \alpha^n (w_0 - \beta w_{-1}) \tag{3.11}$$

for $n \in \mathbb{N}_0$. Hence, from (3.11), we obtain the solution

$$w_{n-1} = \beta^n w_{-1} + (w_0 - \beta w_{-1}) \sum_{i=0}^{n-1} \beta^{n-i-1}\alpha^i$$

or equally

$$w_{n-1} = \beta^n w_{-1} + (w_0 - \beta w_{-1}) \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{3.12}$$

Next, by using (2.3) and the second identity of Lemma 2.1, from (3.12), we have

$$w_{n-1} = F_n w_0 + F_{n-1} w_{-1} \tag{3.13}$$

By returning the original variable x_n via (3.7) and (3.13), we find formula (3.3) for the solution of equation (1.1). Also note that the formula (3.3) is satisfied by α and β which are the fixed points of equation (1.1). That is, by Lemma 2.1, we have

$$x_n = \alpha = \frac{\alpha^{n+1}}{\alpha^n} = \frac{F_{n+1}\alpha + F_n}{F_n\alpha + F_{n-1}}$$

and

$$x_n = \beta = \frac{\beta^{n+1}}{\beta^n} = \frac{F_{n+1}\beta + F_n}{F_n\beta + F_{n-1}}.$$

Now we apply this result to other equations via (1.5). Note that equation (1.1) is transformed by $x_n = -y_n$ into (1.2). From (3.3), by using (2.4), we accordingly get formula (3.4). Similarly, we have formula (3.5) by means of $x_n = \frac{1}{u_n}$. Consequently, since equation (1.1) is transformed by $x_n = -\frac{1}{v_n}$ into (1.4), from (3.3), by using (2.4), we get formula (3.6). The following theorem states the asymptotic behaviors of the solutions of equations (1.1)-(1.4).

Theorem 3.3 The following statements are hold:

a) For $x_0 \in \mathbb{R} - \left(\{\beta\} \cup \mathcal{F}_1\right)$, the every solution of equation (1.1) converge to α, (the golden ratio). That is, $\lim_{n\to\infty} x_n = \alpha$.

b) For $y_0 \in \mathbb{R} - \left(\{-\beta\} \cup \mathcal{F}_2\right)$, the every solution of equation (1.2) converge to $-\alpha$. That is, $\lim_{n\to\infty} y_n = -\alpha$.

c) For $u_0 \in \mathbb{R} - \left(\{-\alpha\} \cup \mathcal{F}_3\right)$, the every solution of equation (1.3) converge to $-\beta$. That is, $\lim_{n\to\infty} u_n = -\beta$.

d) For $v_0 \in \mathbb{R} - \left(\{\alpha\} \cup \mathcal{F}_4\right)$, the every solution of equation (1.4) converge to β. That is, $\lim_{n\to\infty} v_n = \beta$.

Proof. To prove (i), we use formula (3.3) of equation (1.1). To prove (ii), (iii) and (iv), same as before, we apply the result of (i) to the others.

a) For $x_0 \in \mathbb{R} - \left(\{\beta\} \cup \mathcal{F}_1\right)$, from formula (3.3), we can write

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{F_{n+1}x_0 + F_n}{F_n x_0 + F_{n-1}} \\
&= \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \frac{\frac{F_{n+1}}{F_n} x_0 + 1}{\frac{F_n}{F_{n-1}} x_0 + 1} \\
&= \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \frac{\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n} x_0 + 1\right)}{\lim_{n\to\infty}\left(\frac{F_n}{F_{n-1}} x_0 + 1\right)}.
\end{aligned}$$

From (2.5), we get $\lim_{n\to\infty} x_n = \alpha \dfrac{\alpha x_0 + 1}{\alpha x_0 + 1} = \alpha.$

b) Since $x_n = -y_n$, while $x_n \to \alpha$ as $n \to \infty$, for $y_0 \in \mathbb{R} - \left(\{-\beta\} \cup \mathcal{F}_2\right)$, $y_n \to -\alpha$ as $n \to \infty$.

c) Since $x_n = \dfrac{1}{u_n}$, while $x_n \to \alpha$ as $n \to \infty$, for $u_0 \in \mathbb{R} - \left(\{-\alpha\} \cup \mathcal{F}_3\right)$, $u_n \to \dfrac{1}{\alpha} = -\beta$ as $n \to \infty$.

d) Since $x_n = -\dfrac{1}{v_n}$, while $x_n \to \alpha$ as $n \to \infty$, for $v_0 \in \mathbb{R} - \left(\{\alpha\} \cup \mathcal{F}_4\right)$, $v_n \to -\dfrac{1}{\alpha} = \beta$ as $n \to \infty$.

So the proof is terminated.

4. Conclusion

In this study we see that equations (1.1)-(1.4) is transformed into each other by means of relations of (1.5). This case makes possible that if the solution of one of equations (1.1)-(1.4) is investigated then studying of the solution of the others is very easy. Also, the fixed points of these equations can be calculated easily as $\overline{x}_+ = \alpha$, $\overline{x}_- = \beta$, $\overline{y}_+ = -\beta$, $\overline{y}_- = -\alpha$, $\overline{u}_+ = -\beta$, $\overline{u}_- = -\alpha$, $\overline{v}_+ = \alpha$, $\overline{v}_- = \beta$. Hence we can say that while equations (1.1) and (1.3) converge their own positive fixed points, equations (1.2) and (1.4) converge their own negative fixed points. We finally note that the results in this paper are given in terms of Fibonacci numbers.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York (1992).

[2] T. Aboutaleb Mona, M.A., El-Sayed and A. E. Hamza, Stability of the recursive sequence $x_{n+1}=\frac{\alpha-\beta x_n}{\gamma+x_{n-1}}$, J Math Anal Appl. 261, 2001, pp. 126-133.

[3] A. M. Amleh, E. A. Grove, G.Ladas and D. A. Georgiou, On the recursive sequence $y_{n+1}=\alpha+\frac{y_{n-1}}{y_n}$, Math. Anal. Appl. 233, 1999, pp. 790-798.

[4] G. Papaschinopouls and B.K. Papadopoulos, On the Fuzzy Difference Equation $x_{n+1}=A+\frac{B}{x_n}$, Soft Computing 6, 2002, pp. 456-461.

[5] C. Cinar, On the positive solutions of the difference equation $x_{n+1}=\frac{x_{n-1}}{1+x_n x_{n-1}}$, Applied Mathematics and Computation 150, 2004, pp. 21-24.

[6] S. Stević, More on a rational recurrence relation, Appl. Math. E-Notes 4, 2004, pp. 80-84.

[7] M. Aloqeili, Dynamics of a kth order rational difference equation, Applied Mathematics and Computation 181, 2006, pp. 1328-1335.

[8] H. Sedaghat, Global behaviours of rational difference equations of orders two and three with quadratic terms, Journal of Difference Equations and Applications 15(3), 2009, pp. 215-224.

[9] E.A. Grove, Y. Kostrov, G. Ladas, and S. W. Schultz, Riccati difference equations with real period-2 coeficients, Commun. Appl. Nonlinear Anal. 14, 2007, pp. 33-56.

[10] D. Simsek, B. Demir, C. Cinar, On the solutions of the system of difference equations $x_{n+1}=\max\left\{\frac{A}{x_n},\frac{y_n}{x_n}\right\}$, $y_{n+1}=\max\left\{\frac{A}{y_n},\frac{x_n}{y_n}\right\}$, Discrete Dynamics in Nature and Society, 2009, Article ID 325296,11 pages.

[11] E.M. Elabbasy, H.A El-Metwally, and E.M Elsayed, Global behavior of the solutions of some difference equations, Advances in Difference Equations 28, 2011, pp. 2011-28.

[12] E. M. Elsayed, Solution and Attractivity for a Rational Recursive Sequence, Discrete Dynamics in Nature and Society, 2011, Article ID 982309.

[13] D. T. Tollu, Y. Yazlik, N. Taskara, On the Solutions of two special types of Riccati Difference Equation via Fibonacci Numbers, Advances in Difference Equations, 2013, 2013:174.

[14] I. Yalcinkaya, On the difference equation $x_{n+1}=\alpha+\frac{x_{n-m}}{x_n^k}$, Discrete Dynamics in Nature and Society, 2008, Article ID 805460, 8 pages.

[15] N. Taskara, K. Uslu and D.T. Tollu, The periodicity and solutions of the rational difference equation with periodic coefficients, Computers & Mathematics with Applications 62, 2011, pp. 1807-1813.

[16] Y. Yazlik, N. Taskara, A note on generalized k-Horadam sequence, Computers & Mathematics with Applications 63, 2012, pp. 36-41.

[17] K. Uslu, N. Taskara, H. H. Gulec, Combinatorial Sums of Generalized Fibonacci and Lucas Numbers, Ars Combinatoria 99, 2011, pp. 139-147.

[18] S. Falcon, A. Plaza, The metallic ratios as limits of complex valued transformations, Chaos, Solutions & Fractals 41, 2009, pp. 1-13.

[19] El Naschie MS., The golden mean in quantum geometry, Knot theory and related topics, Chaos, Solutions & Fractals 10(8), 1999, pp.1303-1307.

[20] Marek-Crnjac L., On the mass spectrum of the elementary particles of the standard model using El Naschie's golden field theory, Chaos, Solutions & Fractals 15(4), 2003, pp. 611-618.

[21] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley, New York, 2001.