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(m+1)-Dimensional Timelike Parallel p_i-Equidistant Ruled Surfaces with a Timelike Base Curve in the Minkowski Space R_1^n

Melek Masal[*a]

[a] *Sakarya University, Faculty of Education, 54300 Hendek / SAKARYA/ TURKEY,*

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ABSTRACT

The purpose of this paper are to generalize the timelike parallel p_i-equidistant ruled surfaces with a timelike base curve given in Minkowski 3-space R_1^3 ,to generalize n-dimensional Minkowski space R_1^n and to present some characteristic results related with curvatures of the (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in the Minkowski space R_1^n are obtained.

1. Introduction

Firstly, I.E. Valeontis [9], had defined parallel p-equidistant ruled surfaces in and had given some results related with striction curves of these surfaces. Then (m+1)-dimensional ruled surfaces have been studied in n-dimensional Euclidean Space and in Minkowski Space, [1,3,4,7,8].

The purpose of this paper are to generalize the timelike parallel -equidistant ruled surfaces with a timelike base curve given in Minkowski 3-space , [5], to n-dimensional Minkowski space and to present some characteristic results related with curvatures of the (m+1)-dimensional timelike parallel -equidistant ruled surfaces with a timelike base curve in the Minkowski space are obtained.

Throughout this paper, we shall assume that all manifolds, maps, vector fields, etc... are differentiable of class C^∞. First of all, we give some properties of a general submanifold M in

[*] Corresponding author: *E-mail*: mmasal@sakarya.edu.tr, (M. Masal).

R_1^n, [6]. Suppose that $\overline{D}$ is the Levi-Civita connection of R_1^n, while D is the Levi-Civita connection of M. If X and Y are vector fields of M and if V is the second fundamental tensor of M, then by decomposing $\overline{D}_X Y$ into a tangent and normal components we find

$$\overline{D}_X Y = D_X Y + V(X,Y). \tag{1}$$

which is called Gauss Equation.

If ξ is a normal vector field on M, we find the Weingarten equation by decomposing $\overline{D}_X \xi$ in a tangent and a normal component as

$$\overline{D}_X \xi = -A_\xi(X) + D_X^\perp \xi. \tag{2}$$

Here A_ξ represents a self-adjoint linear map at each point and $D^\perp$ represents a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation A_ξ for the linear map and the matrix of the linear map, [2]. If X and Y are vector fields of $\chi(M)$ and the metric tensor of R_1^n is denoted by $\langle , \rangle$ we have

$$\langle \overline{D}_X Y, \xi \rangle = < V(X,Y), \xi > = \langle A_\xi X, Y \rangle \tag{3}$$

If $\{\xi_1, \xi_2, ..., \xi_{n-m}\}$ constitutes an orthonormal base of the normal bundle $\chi^\perp(M)$, we get

$$V(X,Y) = \sum_{j=1}^{n-m} V_j(X,Y)\xi_j . \tag{4}$$

Let M be an m-dimensional semi-Riemannian manifold in R_1^n and A_ξ be a linear map. If $\xi \in \chi^\perp(M)$ is a normal unit vector at the point $P \in M$, then

$$G(P,\xi) = \det \quad A_\xi \tag{5}$$

is called the Lipschitz-Killing curvature of M at P in the direction ξ.

The mean curvature vector H of M at the point P is given by

$$H = \sum_{j=1}^{n-m} \frac{\operatorname{tr} A_{\xi_j}}{\dim \ M} \xi_j \tag{6}$$

Here, $\|H\|$ is the mean curvature. If H vanishes at the each point P of M, then M is said to be minimal, [2].

The 4^{th} order covariant tensor field defined by

$$R(X_1, X_2, X_3, X_4) = < X_1, R(X_3, X_4)X_2 >, \ X_i \in \chi(M) \tag{7}$$

is called the Riemannian curvature tensor and its value at a point $P \in M$ is called Riemannian curvature of M at P.

The sectional curvature function is defined by

$$K(X_p, Y_p) = \frac{< R(X_p, Y_p)X_p, Y_p >}{< X_p, X_p >< Y_p, Y_p > - < X_p, Y_p >^2} \tag{8}$$

where X_P, Y_P are the tangent vectors of tangent plane of M at P. Thus, $K(X_P, Y_P)$ is called the sectional curvature of M at P.

The Ricci curvature tensor field S of M is defined by

$$S(X, Y) = \sum_{i=1}^{m} \varepsilon_i \langle R(e_i, X)Y, \ e_i \rangle \tag{9}$$

where $\{e_1, e_2, ..., e_m\}$ is a system of orthonormal base of $T_M(P)$ and

$$\varepsilon_i = \langle e_i, e_i \rangle = \begin{cases} -1, \text{ if } \ e_i \ \text{ timelike} \\ \ \ 1, \text{ if } \ e_i \ \text{ spacelike} \end{cases}.$$

The value of S(X,Y) at $P \in M$ is called the Ricci curvature. The scalar curvature r_{sk} of M is given by

$$r_{sk} = \sum_{i \neq j} K(e_i, e_j) = 2 \sum_{i \langle j} K(e_i, e_j) \tag{10}$$

2. The Curvatures of (m+1)-Dimensional Timelike Parallel p_i -Equidistant Ruled Surfaces with a Timelike Base Curve in Minkowski Space R_1^n

In this Section, (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in Minkowski space R_1^n are defined. Then, the curvatures of these surfaces are obtained.

Let α and $\alpha*$ be two differentiable timelike curves parameterized by arc length in R_1^n and $\{V_1, V_2, \ldots, V_k\}$ and $\{V_1^*, V_2^*, \ldots, V_k^*\}$, $k \le n$, be their Frenet frames at the points $\alpha(t)$ and $\alpha*(t*)$, respectively. Suppose that M and M* are (m+1)-dimensional timelike ruled surfaces with a timelike base curve in the Minkowski space R_1^n and $E_m(t) = \{V_1, V_2, \ldots, V_m\}$ and $E_m(t^*) = \{V_1^*, V_2^*, \ldots, V_m^*\}$, $2 \le m \le k-2$, are their generating spaces. Then M and M* can be parametrically given by:

$$M: X(t, u_1, \ldots, u_m) = \alpha(t) + \sum_{i=1}^{m} u_i V_i(t), \ \text{rank}\{X_t, X_{u_1}, \ldots, X_{u_m}\} = m+1, \tag{11}$$

$$M^*: X^*\left(t^*, u_1^*, \ldots, u_m^*\right) = \alpha^*(t^*) + \sum_{i=1}^{m} u_i^* V_i^*(t^*), \ \text{rank}\left\{X^*_{t^*}, X^*_{u_1^*}, \ldots, X^*_{u_m^*}\right\} = m+1. \qquad (12)$$

Definition 2.1. Let M and M* be (m+1)-dimensional two timelike ruled surfaces with a timelike base curve. Moreover $p_1, p_2, ..., p_{k-1}, p_k$ denotes distances between the (k-1)-dimensional osculator planes

$$Sp\{V_2, V_3, \ldots, V_k\} \text{ and } Sp\{V_2^*, V_3^*, \ldots, V_k^*\},$$

$$Sp\{V_1, V_3, V_4, \ldots, V_{k-1}, V_k\} \text{ and } Sp\{V_1^*, V_3^*, V_4^*, \ldots, V_{k-1}^*, V_k^*\}$$

$$Sp\{V_1, V_2, \ldots, V_{k-3}, V_{k-2}, V_k\} \text{ and } Sp\{V_1^*, V_2^*, \ldots, V_{k-3}^*, V_{k-2}^*, V_k^*\}$$

$$Sp\{V_1, V_2, \ldots, V_{k-2}, V_{k-1}\} \text{ and } Sp\{V_1^*, V_2^*, \ldots, V_{k-2}^*, V_{k-1}^*\}$$

respectively.

If

1) V_1 and V_1^* are parallel,

2) The distances $p_i, 1 \le i \le k$, between the (k-1)-dimensional osculator planes at the corresponding points of α and α^* are constant,

then the pair of timelike ruled surfaces M and M* are called the (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in the Minkowski space R_1^n.

Throughout this paper M and M* will be assumed (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in the Minkowski space R_1^n**.**

From Definition 2.1, we have

$$V_1 = V_1^*$$

Then, we find the Frenet frames $\{V_1, V_2, \ldots, V_k\}$ and $\{V_1^*, V_2^*, \ldots, V_k^*\}$ are equivalent at the corresponding points of α and α^*.

If k_i and k_i^* are the curvatures of α and α^*, respectively, then we can write

$$k_i = \langle V_i', V_{i+1} \rangle \text{ and } k_i^* = \left\langle V_i^{*\prime}, V_{i+1}^* \right\rangle$$

From the Frenet formulas we can give following theorem:

Theorem 2.1. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve.

i) The Frenet frames $\{V_1, V_2, \ldots, V_k\}$ and $\{V_1^*, V_2^*, \ldots, V_k^*\}$ are equivalent at the corresponding points of α and α^*.

ii) For the curvatures k_i and k_i^* of α and α^*, respectively, we obtain

$$k_i^* = \frac{dt}{dt^*} k_i, \ 1 \le i < k$$

If $\alpha\alpha^*$ is a vector with initial point $\alpha(t)$ and ending point $\alpha^*(t^*)$ of the base curves α and α^* of M and M*, respectively. Then the vector $\alpha\alpha^*$ can be written

$$\alpha\alpha^* = a_1V_1 + a_2V_2 + \cdots + a_mV_m + a_{m+1}V_{m+1} + \ldots + a_kV_k\,,\ a_i \in IR\,,\ 1 \le i \le k$$

Then, we get

$$\left\langle \alpha\alpha^*, V_1 \right\rangle = -a_1\,,\ \left\langle \alpha\alpha^*, V_i \right\rangle = a_i\,,\ 2 \le i \le k\,,$$

and

$$p_i = \begin{cases} |-a_1| \ , & i = 1, \\ |a_i| \ , & 2 \le i \le k. \end{cases}$$

Hence, we have

$$\alpha^* = \alpha - p_1V_1 + p_2V_2 + \cdots + p_mV_m + \ldots + p_kV_k\,.$$

Then we can give following theorem:

Theorem 2.2. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve .The relation between the base curves of M and M* is

$$\alpha^* = \alpha - p_1V_1 + p_2V_2 + \cdots + p_mV_m + \ldots + p_kV_k\,.$$

The space $Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m'\right\}$ is called the asymptotic bundle of M with respect to $E_k(t)$ and denoted by A(t).

If A(t) and $A(t^*)$ are asymptotic bundles of M and M*, respectively, then we have

$$A(t) = Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m'\right\}$$

and

$$A(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, V_1^{*\prime}, V_2^{*\prime}, \ldots, V_m^{*\prime}\right\}.$$

The space $Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m', \alpha'\right\}$ is called the tangential bundle of M with respect to $E_k(t)$ and denoted by T(t).

Let T(t) and $T(t^*)$ are the tangential bundles of M and M*, respectively. So the tangential bundles of M and M* are given by

$$T(t) = Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m', \alpha'\right\}$$

and

$$T(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, V_1^{*\prime}, V_2^{*\prime}, \ldots, V_m^{*\prime}, \alpha^{*\prime}\right\}.$$

Using Definition 2.1 and Theorem 2.1, we find

$$A(t) = A(t^*) = T(t) = T(t^*)\,.$$

Then we can give following theorem:

Theorem 2.3. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with timelike base curve. All the asymptotic and tangential bundles of M and M* are equal.

Now, let us find the matrices A_{ξ_j} and $A_{\xi_j^*}, 1 \le j \le n-m-1$, of M and M*, respectively. If we use the equations (11) and (12), we can easily see that

$$X_t = V_1 + \sum_{i=1}^{m} u_i V_i' \,,\; X_{u_1} = V_1, \ldots,\; X_{u_m} = V_m$$

and

$$X^*_{t^*} = V^*_1 + \sum_{i=1}^{m} u_i^* V_i^{*\prime} \,,\;\; X^*_{u_1^*} = V^*_1, \ldots,\; X^*_{u_m^*} = V^*_m$$

giving the orthonormal bases $\{V_1,\ldots,V_{m+1}\}$ and $\{V^*_1,\ldots,V^*_{m+1}\}$ of M and M*, respectively. If we take the orthonormal bases of the normal bundles $M^{\perp}$ and $M^{*\perp}$ as $\{\xi_1,\ldots,\xi_{k-m-1},\ldots,\xi_{n-m-1}\}$ and $\{\xi^*_1,\ldots,\xi^*_{k-m-1},\ldots,\xi^*_{n-m-1}\}$ then we get the orthonormal bases $\{V_1,\ldots,V_{m+1},\xi_1,\ldots,\xi_{k-m-1},\ldots,\xi_{n-m-1}\}$ and $\{V^*_1,\ldots,V^*_{m+1},\xi^*_1,\ldots,\xi^*_{k-m-1},\ldots,\xi^*_{n-m-1}\}$ of R^n_1 at $P \in$ M and $P^* \in$ M*, respectively, where $\xi_i = V_{m+1+i}$ and $\xi^*_i = V^*_{m+1+i}$, $1 \le i \le k-m-1$. If $\bar{D}$, D and D^* are the Levi-Civita connections of R^n_1, M and M*, respectively, then the Weingarten equations are as follows;

$$\begin{aligned} &\bar{D}_{V_1}\xi_j = \sum_{i=1}^{m+1} a^j_{1i} V_i + \sum_{q=1}^{n-m-1} b^j_{1q}\xi_q \qquad , \; 1 \le j \le n-m-1, \\ &\vdots \\ &\bar{D}_{V_{m+1}}\xi_j = \sum_{i=1}^{m+1} a^j_{(m+1)i} V_i + \sum_{q=1}^{n-m-1} b^j_{(m+1)q}\xi_q \;\; , \; 1 \le j \le n-m-1, \end{aligned} \tag{13}$$

Using the equation (13), for the matrix $A_{\xi_j}, 1 \le j \le n-m-1,$ we get

$$A_{\xi_j} = -\begin{bmatrix} a^j_{11} & a^j_{12} & \cdots & a^j_{1(m+1)} \\ \vdots & \vdots & & \vdots \\ a^j_{(m+1)1} & a^j_{(m+1)2} & \cdots & a^j_{(m+1)(m+1)} \end{bmatrix}. \tag{14}$$

Since α is a timelike curve, one can find following equations

$$\begin{aligned} &a^j_{11} = -\left\langle \bar{D}_{V_1}\xi_j, V \right\rangle && \cdots && a^j_{(m+1)1} = -\left\langle \bar{D}_{V_{m+1}}\xi_j, V \right\rangle \\ &a^j_{12} = \left\langle \bar{D}_{V_1}\xi_j, V_2 \right\rangle && \cdots && a^j_{(m+1)2} = \left\langle \bar{D}_{V_{m+1}}\xi_j, V_2 \right\rangle \\ &\quad\vdots && && \quad\vdots \\ &a^j_{1(m+1)} = \left\langle \bar{D}_{V_1}\xi_j, V_{m+1} \right\rangle && \cdots && a^j_{(m+1)(m+1)} = \left\langle \bar{D}_{V_{m+1}}\xi_j, V_{m+1} \right\rangle \end{aligned} \tag{15}$$

From the equations (2), (3), (14) and (15), we obtain

$$A_{\xi_1} = A_{V_{m+2}} = \begin{bmatrix} 0 & \cdots & 0 & k_{m+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and } A_{\xi_j} = 0,\ 2 \le j \le n-m-1. \tag{16}$$

Similarly, if ξ^* is any normal vector field on M*, we can write

$$\overline{D}_{X^*}\xi^* = -A_{\xi^*}(X^*) + D^{*\,\perp}_{X^*}\xi^*.$$

Then there exists the following equalities

$$\begin{aligned} \overline{D}_{V_1^*}\xi_j^* &= \sum_{i=1}^{m+1} c_{1i}^j V_i^* + \sum_{q=1}^{n-m-1} d_{1q}^j \xi_q^* \qquad , 1 \le j \le n-m-1, \\ &\vdots \\ \overline{D}_{V_{m+1}^*}\xi_j^* &= \sum_{i=1}^{m+1} c_{(m+1)i}^j V_i^* + \sum_{q=1}^{n-m-1} d_{(m+1)q}^j \xi_q^* \quad , 1 \le j \le n-m-1, \end{aligned} \tag{17}$$

Therefore, for the matrix $A_{\xi_j^*}, 1 \le j \le n-m-1$,

$$A_{\xi_j^*} = -\begin{bmatrix} c_{11}^j & \cdots & c_{1(m+1)}^j \\ \vdots & & \vdots \\ c_{(m+1)1}^j & \cdots & c_{(m+1)(m+1)}^j \end{bmatrix}, 1 \le j \le n-m-1 \tag{18}$$

can be written. Since α^* is a timelike curve, we obtain

$$\begin{aligned} c_{11}^j &= -\left\langle \overline{D}_{V_1^*}\xi_j^*, V_1^* \right\rangle & \cdots \quad c_{(m+1)1}^j &= -\left\langle \overline{D}_{V_{m+1}^*}\xi_j^*, V_1^* \right\rangle \\ c_{12}^j &= \left\langle \overline{D}_{V_1^*}\xi_j^*, V_2^* \right\rangle & \cdots \quad c_{(m+1)2}^j &= \left\langle \overline{D}_{V_{m+1}^*}\xi_j^*, V_2^* \right\rangle \\ &\vdots & &\vdots \\ c_{1(m+1)}^j &= \left\langle \overline{D}_{V_1^*}\xi_j^*, V_{m+1}^* \right\rangle & \cdots \quad c_{(m+1)(m+1)}^j &= \left\langle \overline{D}_{V_{m+1}^*}\xi_j^*, V_{m+1}^* \right\rangle \end{aligned} \tag{19}$$

From the equations (2), (3), (18) and (19), we see that

$$A_{\xi_1^*} = A_{V_{m+2}^*} = \begin{bmatrix} 0 & \cdots & 0 & k_{m+1}^* \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and } A_{\xi_j^*} = 0,\ 2 \le j \le n-m-1. \tag{20}$$

Using the Theorem 2.1.ii, we find

$$A_{\xi_1^*} = \frac{dt}{dt^*} A_{\xi_1},\ A_{\xi_j^*} = A_{\xi_j} = 0,\ 2 \le j \le n-m-1.$$

Therefore we have the following theorem.

Theorem 2.4. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in R_1^n. For the matrices A_{ξ_j} and $A_{\xi_j^*}$, $1 \le j \le n-m-1$, we get

$$A_{\xi_1^*} = \frac{dt}{dt^*} A_{\xi_1} \ , \ A_{\xi_j^*} = A_{\xi_j} = 0 \, , \ 2 \le j \le n-m-1.$$

Considering the definition of the Lipschitz-Killing curvature in the direction of ξ_j is found to be

$$G(P, \xi_j) = \det \ A_{\xi_j} = 0 \quad \text{ for all } P \in M, \ 1 \le j \le n-m-1.$$

Similarly, the Lipschitz-Killing curvature in the direction of ξ_j^* of M*, becomes

$$G(P^*, \xi_j^*) = \det A_{\xi_j^*} = 0 \, , 1 \le j \le n-m-1, \text{ for all } \ P^* \in M^*.$$

If H and H* are the mean curvature vectors of M and M*, then from the equations (16) and (20) we have

$$H = H^* = \sum_{j=1}^{n-m-1} \frac{\operatorname{tr} A_{\xi_i}}{\dim M} \xi_j = 0 \, .$$

Therefore, following theorems can be given.

Theorem 2.5. Assume that M and M* are (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curves in R_1^n. The Lipschitz-Killing curvatures of M and M* in all normal directions vanish.

Theorem 2.6. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in R_1^n. M and M* are minimal ruled surfaces**.**

Now, let us find the second fundamental forms, Riemannian curvatures, skalar curvatures and Ricci curvatures of (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in Minkowski space R_1^n**.**

If X and Y are vector fields and V is the second fundamental form of M, then from the equations (3) and (4), we obtain

$$V(X, Y) = -\sum_{j=1}^{n-m-1} < Y, \overline{D}_X \xi_j > \xi_j \, .$$

Thus, for the Frenet vectors V_i and V_j, $1 \le i, j \le m+1$, we have

$$V(V_i, V_j) = -\sum_{s=1}^{n-m-1} < V_j, \overline{D}_{V_i} \xi_s > \xi_s \, , \ 1 \le i, j \le m+1 \, .$$

Then, from the equation (13), we get

$$V(V_i, V_j) = -\sum_{s=1}^{n-m-1} \varepsilon_j a_{ij}^s \xi_s \, , \ \varepsilon_j = \left\langle V_j, V_j \right\rangle = \begin{cases} -1, & j = 1 \\ +1, & j \neq 1 \end{cases}.$$

Using the equation (16), we give

$$V(V_1,V_{m+1}) = -\sum_{s=1}^{n-m-1} a_{1(m+1)}^{s}\xi_s = k_{m+1}V_{m+2} \ ,$$
$$V(V_i,V_j) = -\sum_{s=1}^{n-m-1} \varepsilon_j a_{ij}^{s}\xi_s = 0 \quad , \quad 1 \le i, j \le m+1 \ . \tag{21}$$

Similarly, if X^* and Y^* are vector fields and V^* is the second fundamental form of M*, then from the equation (2) and (3), we have

$$< \overline{D}_{X^*} Y^*, \xi^* > = < V^*(X^*, Y^*), \xi^* > = < A_{\xi^*}(X^*), Y^* > \ , \quad \xi^* \in M^{*\perp}$$

and

$$V^*(X^*, Y^*) = -\sum_{j=1}^{n-m-1} < Y^*, \overline{D}_{X^*}\xi_j^* > \xi_j^* \ .$$

For the Frenet vectors V_i^* and V_j^*, $1 \le i, j \le m+1$, we can write

$$V^*(V_i^*, V_j^*) = -\sum_{s=1}^{n-m-1} < V_j^*, \overline{D}_{V_i^*}\xi_s^* > \xi_s^* \ , \ 1 \le i, j \le m+1$$

and from the equation (17), we obtain

$$V^*(V_i^*, V_j^*) = -\sum_{s=1}^{n-m-1} \varepsilon_j c_{ij}^{s}\xi_s^* \ , \ 1 \le i, j \le m+1.$$

Using the equation (20), we find

$$V^*(V_1^*,V_{m+1}^*) = k_{m+1}^* V_{m+2}^* \ ,$$
$$V^*(V_i^*,V_j^*) = 0 \quad , \quad 1 \le i, j \le m+1 \tag{22}$$

From Theorem 2.1, we have

$$V^*(V_1^*,V_{m+1}^*) = \frac{dt}{dt^*} V(V_1,V_{m+1}) \ ,$$
$$V^*(V_i^*,V_j^*) = V(V_i,V_j) = 0 \quad , \quad 1 \le i, j \le m+1 \tag{23}$$

Hence, we can give following theorem.

Theorem 2.7. Assume that M and M* are (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in R_1^n. V_1 and V_{m+1} are conjugate vectors if V_1^* and V_{m+1}^* are conjugate vectors.

Let $X = \sum_{i=1}^{m+1} a_i V_i$, $Y = \sum_{i=1}^{m+1} b_i V_i \in$ M. Since

$$V(X, Y) = \sum_{i,j=1}^{m+1} a_i b_j V(V_i, V_j)$$

from the equation (21), we have

$$V(X,Y)=a_1b_{m+1}V(V_1,V_{m+1}).$$

Hence, we can give followings.

Theorem 2.8. Let M and M* be (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in R_1^n. Also $X=\sum_{i=1}^{m+1}a_iV_i, Y=\sum_{i=1}^{m+1}b_iV_i\in$M. M is totally geodesic iff $V(V_1,V_{m+1})=0$ or $a_1b_{m+1}=0$.

Corollary 2.1. If $a_1b_{m+1}\neq 0$ and M is totally geodesic, then M* is totally geodesic.

From the equations (7) and (8), we have

$$K(X_P,Y_P)=\frac{\langle V(X_P,Y_P),V(X_P,Y_P\rangle-\langle V(X_P,X_P),V(Y_P,Y_P)\rangle}{\langle X_P,X_P\rangle\langle Y_P,Y_P\rangle-\langle X_P,Y_P\rangle^2}$$

From the last equation and equation (21), we find

$$K(V_1,V_{m+1})=(k_{m+1})^2 \quad \text{and} \quad K(V_i,V_j)=0,\ 1\le i,j\le m+1,\ i\neq j. \tag{24}$$

Similarly, the sectional curvatures of M* are as follows

$$K(V_1^*,V_{m+1}^*)=(k_{m+1})^2 \quad \text{and} \quad K(V_i^*,V_j^*)=0,\ 1\le i,j\le m+1,\ i\neq j. \tag{25}$$

From Theorem 2.1, we can give following theorem.

Theorem 2.9. Suppose that M and M* are (m+1)-dimensional timelike parallel p_i-equidistant ruled surfaces with a timelike base curve in R_1^n. For the Riemannian curvatures of M and M*

$$K(V_1^*,V_{m+1}^*)=\left(\frac{dt}{dt^*}\right)^2K(V_1,V_{m+1}) ,$$

$$K(V_i^*,V_j^*)=K(V_i,V_j)=0 \quad , \quad 1\le i,j\le m+1 \ , \quad i\neq j$$

holds.

The Ricci curvature in the direction V_i of M, is given by

$$S(V_i,V_i)=\sum_{j=1}^{m+1}\varepsilon_j\left\langle R(V_j,V_i)V_i,V_j\right\rangle \ , \quad \varepsilon_j=\left\langle V_j,V_j\right\rangle , \quad 1\le i\le m+1$$

that is

$$S(V_i,V_i)=\sum_{j=1}^{m+1}\varepsilon_j\left\{<V(V_j,V_i),V(V_j,V_i)>-<V(V_j,V_j),V(V_i,V_i)>\right\}.$$

If we use the equation (21), we obtain

$$S(V_{m+1},V_{m+1})=-(k_{m+1})^2 \quad \text{and} \quad S(V_i,V_i)=0 \ , \ 1\le i\le m. \tag{26}$$

The scalar curvature of M is

$$r_{sk} = \sum_{i\neq j} K(V_i, V_j) = 2\sum_{i\langle j} K(V_i, V_j).$$

From the equation (24), we get

$$r_{sk} = 2K(V_1, V_{m+1}) = -2(k_{m+1})^2 . \tag{27}$$

Using the equation (26) we see that

$$r_{sk} = 2S(V_{m+1}, V_{m+1}) \ . \tag{28}$$

Similarly, for the Ricci curvature in the direction V_i^* of M*, we have

$$S(V_{m+1}^*, V_{m+1}^*) = -(k_{m+1}^*)^2 \ \text{ and } \ S(V_i^*, V_i^*) = 0 \ , \ 1 \le i \le m . \tag{29}$$

Moreover, for the scalar curvature of M*, we find

$$r_{sk}^* = 2K(V_1^*, V_{m+1}^*) = 2S(V_{m+1}^*, V_{m+1}^*) \ . \tag{30}$$

From Theorem 2.1, we get

$$S(V_{m+1}^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1}) \ \text{ and } \ r_{sk}^* = \left(\frac{dt}{dt^*}\right)^2 r_{sk} .$$

Hence, we can give following theorem.

Theorem 2.10: Let M and M* be (m+1)-dimensional timelike parallel p_i -equidistant ruled surfaces with a timelike base curve in R_1^n. If $S(V_i, V_i)$ and r_{sk} ($S(V_i^*, V_i^*)$ and r_{sk}^*) are the Ricci and the scalar curvatures of M (M*), then we have

$$S(V_i^*, V_i^*) = S(V_i, V_i) = 0, \ 1 \le i \le m,$$

$$S(V_{m+1}^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1}),$$

and

$$r_{sk}^* = \left(\frac{dt}{dt^*}\right)^2 r_{sk} .$$

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