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On the Elliptic Linear Spacelike Weingarten Surfaces in E_1^3

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ABSTRACT

In this paper, firstly, we get an elliptic linear spacelike Weingarten surfaces. Then we obtain some geometrical properties of elliptic linear spacelike Weingarten surfaces, particularly those related with its Gauss map N.

1. Introduction

A Weingarten (or $W-$) surface is a surface on which there exists a relationship between the principal curvatures. Let f and g be smooth functions on a surface M in Minkowski 3-space. The Jacobi function $\Phi(f,g)$ formed with f, g is defined by $\Phi(f,g)=\det\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ where $f_u=\frac{\partial f}{\partial u}$ and $f_v=\frac{\partial f}{\partial u}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H)=0$ with respect to the Gaussian curvature K and the mean curvature H is called a Weingarten surface or $W-$ surface. Some geometers have studied Weingarten surfaces and obtained many interesting results in both Euclidean and Minkowskian spaces [2, 4, 5, 9, 10, 11, 12, 16, 17, 22, 23].

Linear Weingarten surfaces in R^3 have been also studied by many authors [6,7,13,20]. Chern and Hopf studied the case of the closed surface S with particularly a reference to linearity condition [7, 15]. In this sense, Rosenberg-Earp considered the case of S to be immersed [20]. The surfaces are called H-surfaces, if they have constant mean curvature H

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and are called K-surfaces, if they have constant Gaussian curvature K in Minkowski space. In 1853, Bonnet remarked that the study of K-surfaces could be as difficult as the study of H-surfaces [3].

The two principal curvatures k_1, k_2 satisfy the following equations:

$$K = k_1 k_2$$
$$H = k_1 + k_2$$

and therefore $k_i = H \mp \sqrt{H^2 - K}$, (i=1, 2) [15].

Let S be an orientable surface and $\psi : S \to R_1^3$ be an immersion with Gauss map $\eta : S \to S^2$. It is said that ψ is a linear Weingarten immersion if there exist three real numbers a, b, c not all zero positive real numbers, such that

$$2aH + bK = c$$

where H and K are the mean curvature and the Gaussian curvature, respectively.

In such case we say that S is a linear Weingarten surface where $\sigma = aI + bII$ is a positive definite metric [13]. The equation above is elliptic only when $a^2 + bc > 0$ [15].

In this paper, we consider Weingarten surfaces satisfying elliptic linearity condition such that $b^2 - 4ac < 0$ for spacelike surfaces. Then we give some differential-geometric properties of these kind of surfaces. By using the conformal structure induced by $a\psi - b\eta$, we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map in Theorem 3.1 for elliptic linear spacelike Weingarten surfaces. Throughout this study, the elliptic linear spacelike Weingarten surfaces will be abbreviated as ELSW.

2. Preliminaries

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space R^3 with the metric

$$\langle dx, dx \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

where $(x_1,\ x_2,\ x_3)$ denotes the canonical coordinates in R^3. An arbitrary vector $\mathbf{x}$ of E_1^3 is said to be spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = 0$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq 0$. A timelike or light-like vector in E_1^3 is said to be causal. For $\mathbf{x} \in E_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, then the vector x is called a spacelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = -1$. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [19]. For any two vectors $\mathbf{x} = (x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_1,\ \mathrm{y}_2,\ \mathrm{y}_3)$ of E_1^3, the inner product is the real number $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$ and the vector product is defined by $\mathbf{x} \Lambda \mathbf{y} = \left((x_2 y_3 - x_3 y_2),\ (x_3 y_1 - x_1 y_3),\ (x_1 y_2 - x_2 y_1) \right)$ [18].

A surface in the Minkowski 3-space E_1^3 is called a spacelike surface if the induced metric on the surface is a Riemannian metric. This is equivalent to saying that the normal vector on the spacelike surface is a timelike vector [1].

Let $\{X_u, X_v\}$ be a local base of the tangent plane at each point. Let us recall that the first fundamental form is the metric on $T_P(M)$, that is, the differentiable functions E, F, G: U $\rightarrow$ R

$$E = \langle X_u, X_u \rangle,\ F = \langle X_u, X_v \rangle,\ G = \langle X_v, X_v \rangle$$

are called the coefficients of the first fundamental form I. So the first fundamental form is

$$I = Edu^2 + 2Fdudv + Gdv^2$$

The following differentiable functions

$$e = -\langle X_u, N_u \rangle = \langle N, X_{uu} \rangle$$
$$f = -\langle X_u, N_u \rangle = -\langle X_v, N_u \rangle = \langle N, X_{uv} \rangle$$
$$g = -\langle X_v, N_v \rangle = \langle N, X_{vv} \rangle$$

are called the coefficients of the second fundamental form II. So the second fundamental form is

$$II = edu^2 + 2fdudv + gdv^2$$

[18]. Isothermic parameters are u, v which satisfy

$$ds^2 = E(du^2 + dv^2)$$

[15].

2.1 Theorem A regular surface $M \subset R^3$ is orientable if and only if there is a continuous map $P \rightarrow U(P)$ that assigns to each $P \in M$ a unit normal vector $U(P) \in M_P^{\perp}$ [14].

2.1 Definition Let $S \subset R^3$ be a surface with an orientation η. The map $\eta : S \rightarrow R^3$ takes its values in the unit sphere $S^2 = \{(x, y, z) \in R^3;\ x^2 + y^2 + z^2 = 1\}$. The map $\eta : S \rightarrow S^2$, thus defined, is called the Gauss map of S [8].

2.2 Theorem (Green Theorem) Let P, Q, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ be single-valued and continuous in a simply-connected region R bounded by a simple closed curve C. Then

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

where $\oint_C$ is used to emphasize that C is closed and that it is described in the positive direction [21].

2.2 Definition Let M be a surface in Minkowski 3-space and D be the Levi-Civita connection on Minkowski 3-space. Then,

$$S:\chi(M)\to\chi(M)$$
$$X\to S(X)=D_X N$$

is called the shape operator, where N is the unit normal vector on M [19].

2.3 Definition The Laplacian Δf of a function $f\in\mathcal{F}(M)$ is the divergence of its gradient: Δf =div(gradf) $\in\mathcal{F}(M)$ [19]. A form w is said to be harmonic if $\Delta w=0$ [24].

2.4 Definition A point P of $M\subset\overline{M}$ is umbilic provided there is a normal vector $z\in T_P(M)^{\perp}$ such that

$$II(v,w)=\langle v,w\rangle z$$

for all $v,w\in T_P(M)$. Then z is called the normal curvature vector of M at P [19].

3. On the elliptic linear spacelike Weingarten surfaces in E_1^3

Let S be an orientable spacelike surface and $\psi:S\to R_1^3$ be an immersion with the Gauss map $\eta:S\to S^2$ for S surface. It is said that ψ is a linear Weingarten immersion if there exist three real numbers a, b, c, not all zero, such that

$$2(-a)H+(-b)K=c \tag{1}$$

where H and K mean curvature and Gauss curvature, respectively. The above equation is elliptic only when $a^2-bc<0$ for spacelike Weingarten surfaces. In that case we are going to say that the immersion ψ is elliptic linear spacelike Weingarten (ELSW) surface.

Some interesting examples of ELSW immersions are given by the spacelike surfaces with constant mean curvature, that is b=0, and the spacelike surfaces with positive constant Gaussian curvature, that is a=0.

Lemma 3.1. Let $\psi:S\to R_1^3$ be an ELSW immersion satisfying (1) then there exists a Gauss map $\eta:S\to S^2$ and two real numbers α, β such that

$$2(-\alpha)H+(-\beta)K=\gamma\geq 0 \tag{2}$$

and $\sigma=(-\alpha)I+(-\beta)II$ is negative definite metric, where I=$\langle d\psi,d\psi\rangle$ and II=$\langle -d\eta,d\psi\rangle$ the first and second fundamental form of the immersion, respectively.

Proof If $\{X_1,X_2\}$ is a spacelike orthonormal basis at a point P which diagonalizes $d\eta$, that is, $d\eta(X_i)=-k_i(X_i)$, where $i=1,\ 2$, we have

$$\sigma(X_1\Lambda X_2, X_1\Lambda X_2) = -\left(-(\alpha+\beta k_1)\right)\left(-(\alpha+\beta k_2)\right)$$
$$= -\left(\alpha^2+\beta\left(\alpha(-2H)+\beta(-K)\right)\right)$$

and then

$$\sigma(X_1\Lambda X_2, X_1\Lambda X_2) = -\left(\alpha^2+\beta\gamma\right)<0$$

with $\sigma=(-\alpha)I+(-\beta)II$, that is, σ is negative definite.

Thus, we will assume that every ELSW immersion satisfies the above result. Moreover, Gauss map η given by Lemma 3.1. will be called its associated Gauss map.

Let's obtain a condition in order to define associated Gauss map of ELSW immersion in Lemma 3.2.

Lemma 3.2. Let $\psi: S \to R_1^3$ be an ELSW immersion satisfying the equation (2) with associated Gauss map $\eta: S \to S^2$, then at a point P with Gauss curvature K(P)<0 we have that η(P) is the inner normal if and only if $\alpha \le 0$ or $\beta \le 0$.

Proof If η is not the inner normal at P then the principal curvatures $k_1(P)$, $k_2(P)$ are both negative and using (2), we get

$$k_2 = \frac{\gamma-\alpha k_1}{\alpha+\beta k_1}<0 \tag{3}$$

when $\beta \neq 0$.

Since $\sigma=(-\alpha)I+(-\beta)II$ is negative definite, then

$$\alpha+\beta k_1<0 \tag{4}$$

By using (4) in (3), we have $\gamma-\alpha k_1>0$. Therefore $\alpha k_1<0$. Since $k_1<0$, we have $\alpha>0$ and $\beta>0$. If $\beta=0$, proof of the lemma is obvious. Namely, since η is the inner normal, $\alpha \le 0$ or $\beta \le 0$.

Conversely, let η is the inner normal and $\alpha>0$ or β>0. The following inequality

$$\alpha+\beta k_1<0$$

is satisfied since σ is negative definite. So $\alpha<-\beta k_1$ and since $\alpha>0$ we have $-\beta k_1$>0. As a result, $k_1<0$. Thus $\eta(P)$ is the outer normal. That is contradiction.

Theorem 3.1. Let $\psi: S \to R_1^3$ be an ELSW immersion satisfying the equation (2) with associated Gauss map $\eta: S \to S^2$. Then

$$\Delta^\sigma\psi = \frac{\gamma-\beta K}{\sqrt{\alpha^2-\beta\gamma}}\eta \quad \text{and} \quad \Delta^\sigma\eta = -2\frac{H\gamma+\beta K}{\sqrt{\alpha^2-\beta\gamma}}\eta$$

where Δ^{σ} is the Laplacian operator.

Proof Let (u,v) be isothermal parameters for σ, that is,

$$I = E_1 du^2 + 2F_1 dudv + G_1 dv^2$$
$$II = E_2 du^2 + 2F_2 dudv + G_2 dv^2$$

These equations using in Lemma 3.1., we get

$$\sigma = -\lambda\left(du^2 + dv^2\right)$$

where $\lambda = \alpha E_1 + \beta E_2$

We can write the following formula

$$\begin{aligned} -\alpha\psi_u + \beta\eta_u &= \mu_{11}\ \eta\Lambda\psi_u + \mu_{12}\ \eta\Lambda\psi_v \\ -\alpha\psi_v + \beta\eta_v &= \mu_{21}\ \eta\Lambda\psi_u + \mu_{22}\ \eta\Lambda\psi_v \end{aligned} \tag{5}$$

Where $\eta\Lambda\psi_u$ and $\eta\Lambda\psi_v$ are on the basis of the tangent plane for spacelike surface for certain real numbers $\mu_{11},\ \mu_{12},\ \mu_{21},\ \mu_{22}$. Now, the inner product of these equations with ψ_u and ψ_v, gives

$$\mu_{12} = \frac{\lambda}{\sqrt{F_1^2 - E_1 G_1}} \quad \text{and} \quad \mu_{11} = 0 \tag{6}$$

also

$$\mu_{21} = \frac{-\lambda}{\sqrt{F_1^2 - E_1 G_1}} \quad \text{and} \quad \mu_{22} = 0 \tag{7}$$

Then using the equations (6)-(7) in (5), we get the following formula:

$$\begin{aligned} -\alpha\psi_u + \beta\eta_u &= \frac{\lambda}{\sqrt{F_1^2 - E_1 G_1}}\ \eta\Lambda\psi_v \\ -\alpha\psi_v + \beta\eta_v &= \frac{-\lambda}{\sqrt{F_1^2 - E_1 G_1}}\ \eta\Lambda\psi_u. \end{aligned} \tag{8}$$

By using that

$$\begin{aligned} -\lambda^2 &= \left(-\alpha F_1 - \beta F_2\right)^2 - \left(-\alpha E_1 - \beta E_2\right)\left(-\alpha G_1 - \beta G_2\right) \\ &= \alpha^2\left(F_1^2 - E_1 G_1\right) + \beta\left(2\alpha F_1 F_2 - \alpha E_2 G_1 - \alpha E_1 G_2\right) + \beta^2\left(F_2^2 - E_2 G_2\right), \end{aligned}$$

we get

$$-\lambda^2 = \left(\alpha^2 - \beta\gamma\right)\left(E_1 G_1 - F_1^2\right) \tag{9}$$

Using the equations (9) in (8), we obtain

$$\begin{aligned} -\alpha\psi_u + \beta\eta_u &= \sqrt{\alpha^2 - \beta\gamma}\;\, \eta\Lambda\psi_v \\ -\alpha\psi_v + \beta\eta_v &= -\sqrt{\alpha^2 - \beta\gamma}\;\, \eta\Lambda\psi_u \end{aligned} \tag{10}$$

and

$$\begin{aligned} -\alpha\psi_u\Lambda\eta + \beta\eta_u\Lambda\eta &= \sqrt{\alpha^2 - \beta\gamma}\;\, \psi_v \\ -\alpha\psi_v\Lambda\eta + \beta\eta_v\Lambda\eta &= -\sqrt{\alpha^2 - \beta\gamma}\;\, \psi_u \end{aligned} \tag{11}$$

If we extract the derivative of the first equation with respect to v from the derivative of the second equation with respect to u in (11), we get

$$(-2\alpha H - 2\beta K)(\psi_u\Lambda\psi_v) = \sqrt{\alpha^2 - \beta\gamma}(\psi_{uu} + \psi_{vv}) \tag{12}$$

Using (2) in (12), we obtain

$$(\psi_{uu} + \psi_{vv}) = \frac{\gamma - \beta K}{\sqrt{\alpha^2 - \beta\gamma}}(\psi_u\Lambda\psi_v) \tag{13}$$

On the other hand, if we add the derivative of the first equation with respect to u to the derivative of the second equation with respect to v in (11), we have

$$-\alpha(\psi_{uu} + \psi_{vv}) + \beta(\eta_{uu} + \eta_{vv}) = 2H\sqrt{\alpha^2 - \beta\gamma}(\psi_u\Lambda\psi_v) \tag{14}$$

If we write (13) in (14), we find

$$(\eta_{uu} + \eta_{vv}) = -2\frac{H\gamma + \beta K}{\sqrt{\alpha^2 - \beta\gamma}}(\psi_u\Lambda\psi_v). \qquad (\beta \neq 0)$$

Remark 3.1. Since $H^2 \le K$ on every spacelike surface, given an ELSW immersion satisfying (2) the above inequality occurs when

$$K=\begin{cases} if \quad \beta,\ \gamma\neq 0, & \dfrac{\gamma^2}{\left(\alpha\mp\sqrt{\alpha^2+\beta\gamma}\right)^2} \\ if \quad \beta=0, & \dfrac{\gamma^2}{4\alpha^2} \\ if \quad \beta\neq 0,\ \gamma=0, & \dfrac{4\alpha^2}{\beta^2} \end{cases}$$

As the ELSW surface has negative definite metric, the Gauss map η does not become harmonic. Therefore we cannot obtain the Corollary 3.1. for spacelike surface given in the reference [6].

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